Analyzing strategic interaction in multi-settlement electricity markets: A closed-loop supply function equilibrium model

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Abstract

Multi-settlement electricity markets typically permit firms to bid increasing supply functions (SFs) in each market, rather than only a fixed price or quantity. Klemperer and Meyer's (1989) single-market supply function equilibrium (SFE) model extends to a computable SFE model of a multi-settlement market, that is, a single forward market and a spot market. Spot and forward market supply and demand functions arise endogenously under a closed-loop information structure with rational expectations. The closed-loop assumption implies that in choosing their spot market SFs, firms observe and respond optimally to the forward market outcome. Moreover, firms take the corresponding expected spot market equilibrium into account in constructing their forward market SFs. Subgame-perfect Nash equilibria of the model are characterized

analytically via backward induction. Assuming affine functional forms for the spot market and an equilibrium selection mechanism in the forward market provides for numerical solutions that, using simple empirical benchmarks, select a single subgame-perfect Nash equilibrium.

Incentives for a supplier in the forward market decompose into three distinct effects: a direct effect attributable solely to the forward market, a settlement effect due to forward contract settlement at the expected spot market price, and a strategic effect arising due to the effect of a firm's forward market activity on the anticipated response of the firm's rival. Comparative statics analysis examines the effect of small parameter shocks on the forward market SFs. Shocks that increase the elasticities of equilibrium supply and demand functions tend to make firms more aggressive in the forward market, in that they bid higher quantities at most prices. Expected aggregate welfare for the multi-settlement SFE model is intermediate between that of the single-market SFE model and that of the perfectly competitive case.

Table of Contents

Knowledge is in the end based on acknowledgement.

—Wittgenstein, *On Certainty*

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into the Thompson family in 2000, and offered us occasional refuge from city life at their Vermont home. To my parents, Carolyn and Jerry Anderson, I offer my gratitude and appreciation for their sacrifices and steadfast commitment to my education throughout my life. They instilled in me a work ethic and generosity of spirit that have been instrumental in helping me rise to this challenge. In conclusion, I would not have accomplished this work without the constant love, support, and encouragement of Janet Thompson, my wife. Throughout, Janet labored as hard, if not harder, for the Ph.D. as I did, working several jobs as well as single-handedly managing the quotidian chores of the household. In addition, she pitched in on the thesis itself, bringing to bear her gifts with language and her well-trained editorial eye. She did so cheerfully, and still had it within her to buoy my spirits when things looked bleak. As Goethe observed, “The sum which two married people owe to one another defies calculation. It is an infinite debt, which can only be discharged through all eternity.” I am tempted to add that if one of the pair is writing a thesis then, as the saying goes, pretty soon you’re talking about real money. And so it is to Janet, and the splendid community of our union, that I dedicate this thesis.

To Janet, who always knew

Concern for man himself and his fate must always form the chief interest of all technical endeavors, concern for the great unsolved problems of the organization of labor and the distribution of goods—in order that the creations of our mind shall be a blessing and not a curse to mankind. Never forget this in the midst of your diagrams and equations.

—Einstein, Address at the California Institute of Technology

ELECTRICITY, n. The power that causes all natural phenomena not known to be caused by something else. It is the same thing as lightning, and its famous attempt to strike Dr. Franklin is one of the most picturesque incidents in that great and good man's career. The memory of Dr. Franklin is justly held in great reverence, particularly in France, where a waxen effigy of him was recently on exhibition, bearing the following touching account of his life and services to science:

> Monsieur Franqulin, inventor of electricity. This illustrious savant, after having made several voyages around the world, died on the Sandwich Islands and was devoured by savages, of whom not a single fragment was ever recovered.

—Ambrose Bierce, *The Devil's Dictionary*

It is not from the benevolence of the butcher, the brewer, or the baker, that we expect our dinner, but from their regard to their own interest. We address ourselves, not to their humanity but to their self-love, and never talk to them of our own necessities but of their advantages.

—Adam Smith, *The Wealth of Nations*

1 Introduction

1.1 Electricity sector restructuring

1.1.1 Scope and extent

IN THE 1980S, AND INCREASINGLY IN THE 1990S, dozens of countries around the world initiated economic reforms—or "restructuring"—of their electricity sectors. These countries launched their reforms from widely disparate circumstances, including varied income levels, production and consumption patterns, government roles in the economy, legal and institutional frameworks, and resource endowments. Despite this heterogeneity, the trajectory of electricity restructuring has been broadly similar across countries, typically comprising the following measures (World Energy Council 1998; Girdis 2001):

1. The privatization or corporatization of publicly-owned enterprises in the electricity sector

2. The vertical disintegration, or "unbundling" of the industry's generation, transmission, distribution, and retailing segments
3. The deregulation of the generation and the retailing segments
4. The introduction of regulatory "open access" rules for the transmission segment

In the United States, which has a history of private ownership (though not exclusively so) of electric utilities, restructuring progressed through legislative and regulatory initiatives on two jurisdictional fronts. On the wholesale level, the *Energy Policy Act of 1992* (*EPAct*) catalyzed the development of an open access regime for the electricity transmission grid. Pursuant to this legislation, the Federal Energy Regulatory Commission (FERC)[1] (1996d) issued Order 888, implementing open access and encouraging the formation of independent system operators (ISOs) to manage the transmission grid.[2] Later, the Commission's Order 2000 on Regional Transmission Organizations (RTOs) (Federal Energy Regulatory Commission 1999, 5) laid out an RTO's "minimum" configuration and urged (but did not require) transmission owners to cede control of their transmission facilities to RTOs. In its July 2002 Notice of Proposed Rulemaking (Federal Energy Regulatory Commission 2002a, 3), the Commission built on its earlier initiatives, proposing to establish a standardized transmission service and

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[1] References to "the Commission" throughout this thesis denote the Federal Energy Regulatory Commission.

[2] As of this writing, the Commission had approved the following five ISOs: ISO New England (ISO-NE), California ISO (CAISO), PJM Interconnection (PJM—for portions of the mid-Atlantic states), Midwest ISO (or Midwest Independent Transmission System Operator) (MISO), and New York ISO (NYISO). The Electric Reliability Council of Texas (ERCOT) was created in 1996 by the Public Utility Commission of Texas. ERCOT is contained entirely within the state of Texas, and is hence not subject to the Commission's plenary jurisdiction (Moore and Gooch 2002, 1).

market design to provide a level playing field for all wholesale electricity market participants.

On the retail level,[3] state restructuring initiatives began with a 1993 California regulatory decision (California Public Utilities Commission 1993). As of April 2004, twenty-four states and the District of Columbia had enacted legislation or issued regulatory orders to permit retail access to competitive electricity suppliers; more recently, however, seven of these states delayed or suspended their plans for retail access (American Public Power Association 2004), largely in response to the turmoil in California's market.[4]

1.1.2 Restructuring and economic efficiency

The primary rationale for electricity restructuring in most countries has been to reap welfare gains by supplanting regulation with competition where it is viable. Both theory and experience with other formerly regulated industries suggest that these gains will include increased short-run productive efficiency, enhanced allocative efficiency through pricing that more closely reflects physical and economic reality, and increased dynamic efficiency from improved incentives for investment and innovation. One may gain some perspective on the magnitude of potential efficiency gains for the case of the United States by noting that revenue from electricity sales to final consumers in 2000 totaled approximately $228 billion (Energy Information Administration 2001a, Table A5). By comparison, this amount exceeded recent U.S. annual spending on automobiles,

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[3] While wholesale and interstate transactions are subject to regulation at the federal level by the Commission, retail sales (i.e., sales to final consumers) are under the jurisdiction of each state's public utility commission (PUC) or similar regulatory body.

[4] See Sweeney (2002) and Blumstein (2002) for detailed analyses of the California experience.

telecommunications, or higher education (Brennan et al. 1996, 5). The net book value of electric utility plant owned by major investor- and publicly-owned utilities provides a rough indicator of the size of the industry's total capital stock. As of 1996, this net book value was approximately $433.5 billion.[5]

A potentially significant obstacle to realizing these welfare gains from restructuring is *market power*. Market power exercised by suppliers typically entails the withholding of output and an upward distortion in the market price.[6] Market power is generally associated with various forms of economic inefficiency. Again, it is instructive to consider potential efficiency losses in terms of productive, allocative, and dynamic inefficiencies due to market power. First, market power tends to cause *productive inefficiencies*. To see this, consider a simple example in which a firm—call it "firm A"—exercises market power, restricting its production and driving a wedge between the equilibrium price and its marginal cost.[7] Suppose that firm A's rivals do not exercise market power; they therefore choose their output levels to equate price and their respective marginal costs. In equilibrium, the marginal cost of firm A's rivals exceeds that of firm A, so that aggregate output could be produced at lower total cost if

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[5] As reported by Energy Information Administration (1997, 7) and Energy Information Administration (2001b, Tables 11 and 22) (data for investor-owned utilities were last available for 1996).

[6] Section 1.2 below provides a more formal definition of market power. Market power is usually—but not necessarily—associated with the withholding of output. Hogan (1997) describes a salient exception to this association in an electricity market setting. In a stylized electricity network model with locational marginal pricing, Hogan illustrates how transmission network interactions and constraints enable an owner of generation plants at multiple network locations to exercise market power via *increased* total output. In this event, prices increase at some network locations and *decrease* at others, while total profits for the plant owner increase.

[7] In this stylized example, we assume strictly increasing marginal cost functions and ignore capacity constraints. A profit-maximizing firm, naturally, will always choose its output level to equate marginal revenue and its marginal cost.

production were reallocated from the rivals to firm A. Second, market power also creates *allocative inefficiencies* in that it generally lowers aggregate quantities consumed, causing a deadweight loss to aggregate welfare.[8] Finally, market power creates *dynamic inefficiency* when market participants on both the supply and demand sides of the market make investment decisions based on price expectations distorted by market power. Temporal and (under locational pricing in a transmission network) spatial distortions in prices may arise. Recalling the argument of note 6 above, these spatial pricing distortions due to market power may go in either direction.

Empirical estimates of such welfare losses due to market power have been contentious, but many case studies suggest that such losses have been considerable.[9] Together, the potential magnitude of the problem, controversies surrounding concepts and methodology, and practical difficulties associated with assessing market power underscore the need for substantial further research on this issue. The present investigation constitutes one contribution toward improving the theoretical foundations of market power assessment in electricity markets.

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[8] In the special case of perfectly elastic demand for electricity, there is no loss in allocative efficiency with supply-side market power, rather, only a rent transfer from consumers to producers.

[9] The Department of Energy (2000) reviews empirical research on market power in the United Kingdom, the PJM Interconnection, California, and several other U.S. states. Particularly noteworthy for the present investigation are Borenstein, Bushnell, and Wolak's (2000, 33) findings that from June 1998 through September 1999, electricity suppliers in California's market received revenue in excess of competitive levels of $715 million. These authors later find (Borenstein, Bushnell and Wolak 2002, 1396), moreover, that the problem worsened by the summer of 2000, when (from June to October) the state's electricity suppliers received $4.448 billion in oligopoly rents. In a similar vein, work by the Market Surveillance Committee of the California ISO (2000, 17) found that for May and June 2000, wholesale revenues in the California spot market were 37% and 182% (respectively) in excess of revenues predicted under perfectly competitive pricing.

1.2 Market power

1.2.1 Definition and origins

To economists, market power is "the ability to alter profitably prices away from competitive levels" (Mas-Collel, Whinston and Green 1995, 383).[10] As it relates to industry structure, market power on the part of suppliers is commonly classified as either *horizontal* or *vertical*. Vertical market power is the ability to engage in exclusionary behavior conferred by one's control of different segments of the industry: generation, transmission, distribution, and retail services. Horizontal market power, in contrast, is the ability to influence price within one of these segments.

Historically, most of the world's electricity industries consisted of vertically integrated, publicly-owned and/or -regulated monopolies with exclusive geographic franchises. In the United States, private ownership of electric utilities has been the norm, under which state regulatory commissions established prices consistent with a "just and reasonable" standard (see, e.g., Phillips 1993, 119, ch. 5). Under a competitive regime, in contrast, interactions between competing generating firms would determine prices endogenously. In light of this regulatory legacy, the deregulation of generation would endow these utilities—de facto vertically-integrated regional monopolies—with considerable market power. The advantages of incumbency enjoyed by these monopolies would not necessarily be overcome by the timely entry of new competitors. [11]

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[10] While either suppliers or demanders may possess market power, we consider only supply-side market power in this investigation. See also note 6 above.

[11] It is sometimes argued that entry will significantly lessen concerns over horizontal market power, rendering it at best a transitional problem. In the abstract, this reasoning has some appeal. It is often the case today, however, that formidable entry barriers (e.g., local siting restrictions) for new generation and transmission facilities characterize electricity markets in the United States, particularly close

1.2.2 Policy response

A number of prominent industry observers have argued that federal lawmakers have granted the FERC adequate authority to address vertical market power in the U.S. electricity industry. For example, Pierce (1996, 32) writes that "for antitrust purposes, the FERC can now ignore the vertical constraints on competition that were the primary focus of the FERC's antitrust activities during the 1980s. As amended by the *EPAct*, the FPA [*Federal Power Act*] now gives the FERC regulatory tools that allow it to address . . . [these] vertical constraints."[12] Via its open-access transmission policies (Federal Energy Regulatory Commission 1996d, 1999) the Commission has, in fact, brought these tools to bear on vertical market power concerns. In the future, the Commission's commitment to RTOs may reasonably be expected to mitigate substantially if not eliminate any remaining vertical market power problems. On the state level, moreover, regulators (typically, state attorneys general or PUCs) in many jurisdictions have insisted upon the divestiture of vertically-integrated utilities' generation assets as a quid pro quo for recovery of "stranded costs," or sunk costs in

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to urban centers where electrical load is concentrated. One also commonly observes transmission constraints in such settings, creating so-called "load pockets." Thus, the short-run transition may indeed last for some time, and may well be associated with considerable efficiency losses as well as significant transfers to suppliers with market power.

[12] This assessment has so far proved perhaps too optimistic, since as the Commission wrote later in its Order 2000, "we . . . conclude that opportunities for undue discrimination continue to exist that may not be adequately remedied by functional unbundling [see below]. We further conclude that perceptions of undue discrimination can also impede the development of efficient and competitive electric markets. These concerns . . . provide the basis for issuing [Order 2000]" (Federal Energy Regulatory Commission 1999, 65). Functional unbundling, required by the Commission's Order 888, comprises three restrictions on conduct for a vertically-integrated utility. The utility must (1) take transmission services under the same tariff as do others, (2) post separate rates for generation, transmission, and ancillary services, and (3) rely on the same electronic information network as do its transmission customers when arranging transactions (Federal Energy Regulatory Commission 1996d, 57).

excess of market prices.[13]

As for horizontal market power, Pierce (1996, 32) also remarks that "the FERC needs to refocus its antitrust attention on horizontal market power issues. . . ." Indeed, horizontal market power in the industry's generation segment has emerged as a central public policy concern in U.S. electricity restructuring. For this reason, this investigation focuses exclusively on horizontal market power—denoted hereinafter simply as "market power"—in the electricity industry's generation segment. The following chapter, chapter 2, provides a more detailed account of the policy response in the U.S. to market power.

1.2.3 Motivation and objectives of the present investigation

Below, we detail some gaps and inadequacies in both the theoretical foundations for market power analysis in restructured, competitive electricity markets and in the policy framework for addressing market power problems. In light of the dramatic structural changes in the electricity industry worldwide, the relevant theory needs to be refined and extended. The highly-structured institutional environment that is necessary to coordinate efficiently firms' behavior in electricity markets creates complex incentives; these incentives render the characterization of market power in this context a difficult—and unfinished—task. To advance the discussion, a fruitful starting point would be to lay the analytical foundations for defining and measuring market power given the architecture of today's competitive electricity markets. The present work provides a coherent, if stylized, characterization of key incentives that market participants face in this

[13] While most commentators have taken the view that these regulators lack the authority to compel divestiture, the prospect of (at least partial) denial of stranded cost recovery induced some integrated utilities to divest generation assets. Such divestiture, of course, can have both horizontal as well as vertical competitive ramifications.

environment, which is necessary to guide the development of analytical methodologies for empirical analyses of market power. Policymakers and regulators may then bring such methods to bear in assessing the severity of market power, and in crafting appropriate and welfare-enhancing policy responses.

The present investigation focuses on the competitive implications of a *sequence* of markets, an architectural feature present in many competitive electricity markets. This particular element of market design creates intertemporal incentives for market participants—related, in general, to risk hedging, speculation, and strategic considerations (see, e.g., Allaz 1987)—the effects of which are as yet poorly understood. This thesis examines the behavioral incentives induced by the architecture of newly restructured electricity markets. In particular, we derive profit-maximizing supply function[14] equilibrium bids for electricity suppliers competing in sequential forward and spot markets. In a series of numerical examples, we examine how these bids depend on underlying economic characteristics of this environment, and compute expected aggregate welfare for this market setting. Sections 1.3 and 1.4 below elucidate the scope of this investigation in greater detail.

1.3 Modeling competitive electricity markets

1.3.1 Market characteristics

Competitive electricity markets in the U.S. share some salient market design features with others around the world. Among these common characteristics are the existence of forward markets (in addition to the spot market) for electricity, significant flexibility in

[14] For more on supply functions, see subsection 1.3.3.

the form of bids permitted from suppliers, uncertainty in demand, and determination of prices via a market-clearing competitive equilibrium.

First, the design of many of the world's electricity markets includes at least one—and sometimes several—forward energy markets. Such a market design is commonly referred to as a *multi-settlement market*.[15] When forward energy markets clear close to real time (e.g., one day ahead), they typically rely on a market coordinator and competitors' bids (rather than on bilateral negotiation) to set price. Second, market rules in many electricity markets around the world permit significant flexibility in firms' supply bids, requiring only that bids take the form of increasing functions from price to quantity.[16] Third, demand uncertainty in each periodic market[17] arises from uncertain weather conditions, equipment failure, and other contingencies. Market participants may make demand forecasts to aid their market decision making, but these forecasts will naturally be imperfect. Last, the point at which the aggregate supply function intersects aggregate demand normally determines the market-clearing price in each market. The approach to modeling competitive electricity markets described in the remainder of this section and developed later in the thesis reflects each of these market characteristics.

Other features of electricity markets having significant competitive implications include the interconnected transmission and distribution network, intertemporal

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[15] The modifier "multi-settlement" denotes that the forward and spot markets entail distinct financial settlements (billing and payment) between buyers and sellers in the respective markets. The cash flow paid or received by a participant in a particular market's settlement is, naturally, the product of the market-clearing price and that market participant's quantity bought or sold. See note 29 below for further details.

[16] These functions are sometimes also required to be continuous.

[17] Competitive electricity markets for energy typically comprise regular, periodic spot markets (e.g., hourly or half-hourly) during each day. Associated with each period's spot market may be one or more forward markets as well as markets for reserves (i.e., generating capacity).

constraints, and multiple products and markets. This investigation abstracts, for simplicity, from the complications associated with these characteristics. We discuss briefly below the implications of these simplifying assumptions.

The transmission and distribution network is necessary, of course, for transport and delivery of electricity as well as for ensuring reliability and quality (e.g., voltage and frequency stability). Because network capability is limited, the competitive price of electricity will vary across different locations (in addition to temporal variations) under locational marginal pricing. The model developed here simplifies this situation considerably. It may be interpreted as a model analyzing competition at a single network location. Alternatively, one may view the present work as modeling a completely uncongested transmission network while also ignoring transmission losses.

Electricity generation technologies exhibit to varying degrees numerous dynamic constraints restricting the pattern and associated costs of generation plant production over time. Examples of such constraints include minimum times for startup and shutdown (with associated costs), minimum run times, and ramp rate constraints. Startup costs imply that a currently idle unit may not find it profitable to begin operation in a given hour if expected prices in the near term are insufficient to cover its variable operating cost as well as its startup cost. Ramp rate constraints limit the amount by which a generating unit can change its production level from one hour to the next. In practice, these constraints have potentially significant economic implications for generating units' operating schedules. Proper analysis of such constraints is complicated not only by their intertemporal nature, but also because they introduce non-convexities into the unit's production function. We abstract from all such complications by assuming (see subsection 3.1.8) that cost functions are strictly convex and that there are no intertemporal operating constraints of economic importance. This rather strong assumption permits us to analyze each operating period independently.

Finally, competitive electricity markets comprise multiple products and markets. As one salient example, in the early days of California's restructured market, there was a total of eleven markets for energy and ancillary services.[18] Designs in most other regions do not include as many distinct product markets, although most competitive electricity markets do feature, at a minimum, both forward and real-time (or "spot") energy markets. The present work presumes the simplest market architecture—a single forward market and a spot market—that permits us to examine the influence of multiple markets on competition. Introducing additional product markets (e.g., for ancillary services) would substantially complicate the analysis. An extension of the present model to a sequence of two or more forward markets in advance of the spot market would be relatively straightforward, at least conceptually.

1.3.2 Application of game theory

As a general matter, it is natural to model interactions among agents in diverse market settings using the tools of game theory. This is particularly true in electricity markets, in which market participants' interactions are highly structured and regularized via market institutions—witness the centrally-cleared markets for electrical energy and ancillary services organized by various system operators around the world. Such electricity

[18] "Ancillary services" refer to reserve generation capacity, available on timescales varying from instantaneous to up to several hours. California's original market design envisioned four ancillary services traded in day-ahead and hour-ahead markets and imbalance energy dispatched in real time by the CAISO. The CAISO operated these nine product markets. In addition, the (former) California Power Exchange cleared day-ahead and hour-ahead markets for energy, for a total of eleven product markets.

markets have, in effect, a well-defined set of *players*, and for each player, a *strategy space* and a *payoff function*; these elements of electricity market design are also the basic constituent elements of any game-theoretic model.

In electricity markets, the supply side is sometimes sufficiently concentrated that the presumption of perfectly competitive (i.e., price-taking) behavior on the part of suppliers seems inappropriate.[19] Instead, models that permit oligopolistic interaction—or imperfect competition—are relevant in this context; these models capture the ability of individual producers to influence the market-clearing price. While several alternative models of oligopoly behavior have been widely applied,[20] the features of competitive electricity markets reviewed above strongly suggest that *supply function equilibrium* (SFE) models are best suited for modeling such markets, as the following subsection elaborates.

1.3.3 Supply functions

In the *supply function* (SF) *model* developed in this investigation, players' strategy spaces are the set of strictly increasing continuous functions from price to quantity. Since forward markets such as the former California Power Exchange (PX) commonly require—as with spot markets—that suppliers' bids be increasing continuous functions or step functions, multi-settlement markets (see subsection 1.3.1) lend themselves to a nested SF model in which firms bid SFs in both the forward and the spot markets. For

[19] Apart from concentration, common features of electricity markets generally viewed as contributing to market power are the inability to store electricity economically together with the necessity of instantaneous supply and demand balance at every location in the transmission grid, and demand inelasticity, particularly in the short run.

[20] Well-known oligopoly models include those of Cournot, Bertrand, conjectural variations, and Bertrand-Edgeworth; see Vives (1999) for a comprehensive survey of these models and their application.

each market, the behavioral assumption of SFs—in contrast to pure quantity or price choice under Cournot or Bertrand competition—is suggestive of the range of strategies actually available to suppliers in competitive electricity markets.[21] As discussed further below, the SF model explicitly recognizes and accommodates demand uncertainty: SF bids enable suppliers to achieve an ex post optimal outcome under any realization of uncertain demand.[22] Finally, price formation in SF models occurs consistent with basic economic intuition: the point of intersection of aggregate supply and aggregate demand determines the market-clearing price.

The above discussion suggests that models based on the SF behavioral assumption possess a striking verisimilitude to the characteristics of competitive electricity markets, and that SF models, therefore, are especially well-suited to modeling supplier behavior realistically in such markets. Moreover, we may extend the single-market SFE framework developed in Klemperer and Meyer's (1989) seminal paper to a multi-settlement market. Accordingly, we assume in this work that suppliers bid (strictly increasing) SFs in both the forward and spot markets.

In the single-market SFE framework, firms' equilibrium SF bids will result in ex post optimal production at any market-clearing price. Put another way, for any realization of demand uncertainty, a firm that bids its equilibrium SF given the SFs of the

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[21] The SFE framework is inherently more flexible than Cournot or Bertrand, allowing suppliers to specify through their bids a schedule of quantities over a range of prices, rather than a fixed quantity or price. In this sense, we may view SF-based models as a generalization of the Cournot or Bertrand frameworks. Such flexibility in firms' strategies is present in contexts other than electricity, as well. Namely, Klemperer and Meyer (1989) cite as salient examples the airline industry—in particular, its computerized reservation system—and management consulting.

[22] This strong result is strictly true only in a single-market setting, and must be qualified somewhat in a multi-settlement market context, as we discuss in subsection 3.4.3.

other bidders guarantees that it will be called upon to produce its optimal quantity. Remarkably, a firm's equilibrium SF in the single-market SFE framework is distribution-free, in that it is independent of the (non-degenerate) probability distribution of the uncertain demand shock. This property of SFs may at first appear counterintuitive, but it is attributable precisely to the way in which the SF is constructed. As chapter 4 will show, every feasible value of the stochastic shock to demand corresponds to a distinct point on the corresponding SF.[23]

In contrast, the extent of the SF—that is, the domain of prices over which the SF is defined—does depend on the support of the demand shock. This is a direct consequence of the claim above that every feasible value of the demand shock corresponds to a distinct point on each firm's SF. Moreover, the expected values of price, quantity, and profits associated with a given SF do depend, as intuition would suggest, on the probability distribution of the demand shock.

In the multi-settlement market framework investigated here, this argument must be modified. It is natural, in this setting, to take forward market equilibrium as contingent on the expected outcome in the spot market. Doing so, forward market SFs then depend on the distribution of the uncertain spot market demand shock. The forward market SFs, therefore, no longer possess the distribution-free property exhibited by SFs in a single-market setting. As for spot market SFs, once the forward market has cleared,

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[23] Since we will assume the demand shock to have an atomless distribution, any arbitrary value of the shock occurs with probability zero. By continuity, the values of the stochastic shock within an (arbitrarily small) interval correspond to a particular section of an SF; the probability that the shock takes on a value in this interval is strictly positive. Whether the probability of a realization of the shock in this interval is large or small—that is, the shock's probability distribution—is inconsequential; it matters only that this probability is strictly positive, that is, that such shocks can occur. Given such a shock, firms respond via their SF bids to maximize profits. The SF, therefore, is defined over equilibrium prices corresponding to the shock's entire support, the union of all such feasible intervals.

the spot market is effectively a single market. Thus, as with the single-market SFE, spot market SFs in the multi-settlement market will again be distribution-free.

1.4 A closer look at market power

1.4.1 Competing definitions and the degree of market power

Subsection 1.2.1 appealed to a standard text on microeconomic theory to define market power as "the ability to alter profitably prices away from competitive levels" (Mas-Collel, Whinston and Green 1995, 383); this is the definition that we apply for sellers throughout the present work. Interestingly, federal antitrust regulators—and by reference, the Commission—use a somewhat more restrictive definition of market power, namely, "market power to a seller is the ability profitably to maintain prices above competitive levels for a significant period of time."[24] Stoft (2002, 366) explores the differing implications of these two definitions, and argues (p. 368) that, under either standard, "the goal should never be the prevention of all market power." Rather, regulators inevitably "need to make a hard decision: How much market power is too much?"

Indeed, the question of the degree of market power—under either definition—is central to any welfare-based assessment of market power that would balance efficiency losses due to market power with the direct and indirect costs of market intervention. Note that while both of the above definitions of market power refer to "competitive

[24] Department of Justice (DOJ) and Federal Trade Commission (FTC) (1992, Sec. 0.1) Horizontal Merger Guidelines. The FERC's Merger Policy Statement (1996f) states that the FERC will use the screening approach of the DOJ/FTC Merger Guidelines to determine whether a merger will result in an increase in market power.

[price] levels," neither definition is explicit about what constitutes such levels.[25] Mas-Collel, Whinston, and Green(MWG)'s (1995) discussion permits us to make some conceptual headway, although here, too, we are ultimately left with unanswered theoretical questions.

From MWG (pp. 314–315), we may infer that competitive price levels are those that clear the market in a "competitive economy" which, in turn, is one in which all consumers and producers act as price-takers. MWG elaborate that "[f]or the price-taking assumption to be appropriate, what we want is that [consumers and producers] have no *incentive* to alter prices that, if taken as given, equate demand and supply" (emphasis in original). For the purposes of this investigation of supply-side market power, we then confront two questions:[26]

1. What constitutes price-taking behavior for supply? In other words, what is the appropriate *perfectly competitive behavioral benchmark* (PCBB) for a supplier?
2. What equilibrium price results from such price-taking behavior?

Given an answer to question 1 above, one easily obtains the answer to question 2 by computing the set of prices (not necessarily unique) that clear the market. Thus, question 1 is an interesting and important question for market power analysis. An appropriate PCBB may serve, in particular, as a foundation for empirical work assessing the severity of market power. Namely, by comparing observed bid prices with those simulated using

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[25] Remarkably, the Horizontal Merger Guidelines (Department of Justice and Federal Trade Commission 1992) themselves fail to supply any guidance for what constitutes "competitive [price] levels."

[26] In this investigation, we assume price-taking demand while permitting strategic behavior on the supply side.

the PCBB, we may—subject to the limitations of the particular modeling framework adopted—shed light on the question of whether a supplier has exercised market power.

Intuitively, the PCBB depends on the incentives, and hence the institutional environment, that agents face. In an idealized single-period, bid-based market,[27] we may appeal to basic economic intuition: the PCBB would be a price bid of marginal cost for all quantities up to one's production capacity. In the multi-settlement market setting considered in this thesis, establishing what constitutes the PCBB is a more subtle and complex question. One would generally need to consider (as we do here) the effect of the forward market on spot market behavior as well as firms' anticipation—and thus the influence—of the later spot market equilibrium on their prior forward market behavior.

The principal goal of the present investigation is, therefore, to characterize and analyze the inter-market incentive effects that exist in a multi-settlement market. Achieving a solid understanding of such effects is the first step toward determining a well-founded and internally consistent PCBB, a task that itself is beyond the scope of this work.

1.4.2 Forward contracting and market power assessment

Long-term forward contracts for energy generation had been a common feature of the electricity industry before restructuring, and they continue to play a role in today's more competitive environment. They have been instrumental in providing a secure return on investment, thereby facilitating project financing. Once an investor is committed to a

[27] Ignoring start-up and no-load costs, and any other non-convexities of firms' cost functions.

project, such long-term contracts also help to alleviate the hold-up problem.[28] This subsection outlines in more detail the extension of market power analysis to consider markets for forward contracts.

Short-term (e.g., day-ahead) forward contracts for energy—a more recent financial innovation—trade in centrally-cleared markets organized by most U.S. ISOs. These contracts enable both *hedging* of spot (e.g., real-time) prices for both buyers and sellers, thus reducing risk, and *financial speculation* which enhances liquidity in the short run. If the forward market is a credible price benchmark, this can facilitate development of futures and options markets for electricity in the longer run. These markets, in turn, are likely to narrow spreads in the various markets, and will provide market participants with more flexibility than would long-term bilateral contracts. Assuming a reasonably liquid market, it will be easier and more efficient for market participants to use these financial derivatives rather than to renegotiate a bilateral contract when circumstances change (since contractual counterparties have opposing interests in such renegotiations). A market in short-term contracts can fulfill the additional function of price discovery, allowing market participants to profitably exploit technical flexibility. Contracts also support generator scheduling and unit commitment, providing a baseline for potentially profitable rescheduling (e.g., through "Schedule Adjustment Bids" in the (former) California PX).

Multi-settlement markets—that is, forward and spot markets that clear at distinct points in time—are a common feature of competitive electricity markets around the

[28] The hold-up problem is the ability of opportunistic regulators or a monopsonistic buyer to appropriate the scarcity rents from illiquid fixed assets (e.g., electricity generation plants) once the investment is sunk by permitting spot prices to cover only marginal cost.

world.[29] The following electricity markets feature a multi-settlement market structure (Jamasb and Pollitt 2001, 17–18): Australia (New South Wales, Queensland, Victoria), Canada (Ontario), Colombia, England and Wales, France, Ireland, New Zealand, Nordpool (Finland, Norway, and Sweden), and the United States (PJM, New York, New England, and proposed in the Midwest (Midwest ISO 2004)). The intertemporal character of multi-settlement markets raises the following policy issues regarding design and regulation of these markets:

1. Both ex ante market design and ex post assessments of electricity spot market performance need to take into account (a) how forward contracting changes the expected payoffs from (and hence incentives for) spot market activity, and (b) what these effects imply for the assessment of market power in a multi-settlement market.
2. How may we evaluate the performance of the forward market itself? For example, is there a perfectly competitive behavioral benchmark that applies to the forward market in isolation? Or, does assessing market power in multi-settlement markets require joint evaluation of behavior in forward and spot markets?

Overall, the theoretical foundation for understanding and assessing market power in multi-settlement markets is weak and incomplete. Questions such as these are only

[29] In general, a multi-settlement market is a *sequence* of markets for a product that includes

1. at least one "forward market," in which buyers and sellers may conclude financial contracts for later delivery, and
2. a "spot market," which clears contemporaneously with delivery of the product.

While the approach outlined here could, in principle, be extended to include two or more forward markets, this thesis considers a single period of forward trading preceding the spot market. Market participants may transact in both the forward and spot markets, modifying their forward positions in the later spot market, if they choose. In this thesis, we take forward contracts to be legally binding.

beginning to be addressed by the relevant academic literature (reviewed in section 1.5 below). The practical significance of these lacunae has been particularly acute in the context of California's electricity markets. The original California market design of centrally-organized forward and spot markets for energy (and ancillary services) was an early and salient example of a multi-settlement market. In this environment, market power analyses based on the conventional single-market model have been contentious and a target for criticism. Quan and Michaels (2001, 100), for example, " . . . believe that analyses of the [California] ISO and PX have often reached conclusions about market power on the basis of abstractions that obscure and misinterpret important aspects of competitive behavior."

One early study of the California markets by the Market Monitoring Committee (MMC) of the California Power Exchange (1999) evinces the difficulties to which Quan and Michaels allude. The MMC's study proposed (p. 58) to "assess a firm's perceived market power by calculating the Lerner Index at each quantity level it bids [in the PX's hourly energy auction], and then averaging the Lerner Index values over the whole bid curve. Specifically, for each hour we used the firm's actual bid curve and our estimate of its marginal cost to calculate the weighted average gross margin." Thus, the MMC defines a PCBB for forward market bidding behavior based on marginal production cost. It denotes this as the "Bid-Markup Index (BMI)," defined algebraically as

$$BMI(t) = \int_0^{Max\,Q(t)} \left[1 - \frac{C'(q,t)}{\rho(q,t)}\right] dq\,,$$

where

$BMI(t)$ = Bid-Markup Index for hour t

$Max\ Q(t)$ = Maximum quantity offered at any price at or below $250/MWh in hour t

$\rho(q,t)$ = Bid price at which the firm offers quantity q in hour t

C' = estimated marginal cost of the firm.

Elsewhere in the report, the MMC recognizes the potential importance of opportunity costs introduced by the presence of the later spot market (see, e.g., their discussions on pp. 12–13 and 50–53). Moreover, the MMC is ultimately cautious in drawing conclusions regarding the exercise of market power in this novel and rapidly evolving market environment. The MMC's explicit choice of a marginal cost-based benchmark as a PCBB would be placed on firmer conceptual footing, however, if supported by a formal model.

Borenstein, Bushnell, and Wolak (2002) also study market power within the California markets. They argue that forces of arbitrage across the spot and forward markets will tend to make prices in these markets converge, and find that such price arbitrage is supported by their data.[30] Given these observations, the authors' use of an estimated marginal production cost function for fossil-fuel generation[31] as the PCBB for energy bid into either market is internally consistent. In the present work, we do not assume arbitrage in the sense of Borenstein, Bushnell, and Wolak, but instead take

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[30] Over their sample period of June 1998 to October 2000, "the PX average price was not significantly greater than the ISO average price" (Borenstein, Bushnell and Wolak 2002, 1384). If one also invokes the rational expectations assumption, under which agents (unobserved) ex ante expectations are consistent with ex post realized price distributions, then we may conclude that for their sample period, spot and forward market prices are equal in expectation. See also Borenstein, Bushnell, Knittel, and Wolfram (2001).

[31] The authors focus on residual demand—that is, total demand net of demand met by non-fossil fuel generation—in the market power analysis. The estimated marginal production cost for fossil-fuel generation accounts for generator efficiency and availability, fuel costs, and variable operating and maintenance expenses.

demand to be strictly risk averse[32] (while assuming supply to be risk neutral). Under these more general circumstances, it is no longer clear that marginal production cost is the appropriate PCBB for the forward market.

Multi-settlement markets raise issues of *bidding based on opportunity costs* and *scarcity*, each of which is distinct from market power as defined above. Distinguishing these issues both conceptually and empirically has been the subject of much debate and confusion. In the paragraphs below, we briefly contrast these concepts with market power.

Market power vs. bidding based on opportunity cost. Marginal opportunity cost ("*MOC*") for a firm is the marginal revenue from the highest-valued alternative sales opportunity for an increment of output. In electricity markets, such outside options—that is, alternative market opportunities for a given increment of generating capacity—are the rule rather than the exception. Such possibilities may be due simply to geography, such as the prospect of exporting power outside of a given regional market. Alternatively, the architecture of electricity markets may offer these opportunities, for example, the chance—within a given regional market—to sell ancillary services (see n. 18), instead of selling into a forward energy market. Each such alternative opportunity is associated, at least in principle, with an *MOC*. When such opportunities exist, the conceptually appropriate benchmark for assessing the competitiveness of market behavior (e.g., a firm's SF bids) would be the greater of marginal production cost (*MPC*) and *MOC* (see, e.g., Borenstein, Bushnell and Wolak 2000, 6–7).

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[32] This assumption motivates the derivation (in chapter 6 of the present work) of an endogenous forward market demand function. In our framework, we may model risk neutrality of demand as a limiting case by permitting the parameter capturing demand's risk aversion (see subsection 6.2.2) to approach zero.

In the multi-settlement SFE model, there is a (probabilistic) opportunity cost in making forward market commitments, even though financial contracts may be unwound, or reversed, in the spot market. Such opportunity cost arises because of the chance that a generator might contract forward to sell quantity q^f at a contract price p^f, which may turn out to be less than the later spot price, p^s. In a competitive equilibrium, we should expect this risk to be reflected in contract reservation prices (i.e., forward market bids), both for firms exercising market power as well as for perfectly competitive firms.

Market power vs. scarcity. In a given competitive market equilibrium, the difference between a particular generator's revenue and its total variable costs is commonly referred to as scarcity rent. Scarcity rents contribute to covering generators' fixed costs. They are particularly important for peaking generation capacity, which operate for relatively few hours each year. If sufficient over time, scarcity rents can also provide the necessary incentive for investment—either by existing market participants or new entrants—in new generation, transmission capacity, or demand management technologies. Absent capacity withholding, however, there is no welfare loss associated with the existence of scarcity rents (rather, only a wealth transfer), and therefore, no exercise of market power.

The present model assumes no generation capacity limits, and so will not address the traditional notion of scarcity directly. However, a strictly increasing marginal production cost function—which we do assume—serves, in effect, as a soft capacity constraint: it increases the average opportunity cost (see above) of firms' forward market positions. In this sense, then, scarcity will play a role in the multi-settlement SFE model analyzed here.

1.5 Existing literature

As in other settings, simple models of quantity or price choice under the Cournot or Bertrand frameworks have formed the basis of many studies of electricity market competition. As subsection 1.3.3 explained, we may view the SFE framework as a generalization of these simpler competitive models, and one, in particular, having a greater degree of verisimilitude to the architecture of many competitive electricity markets. Accordingly, in this section's review of relevant literature, we focus primarily on single-settlement SFE models and studies of multi-settlement markets. Kamat and Oren (2002) cite additional relevant sources and provide a useful overview of recent work on market power in competitive electricity markets.

1.5.1 Single-settlement SFE models

Klemperer and Meyer (1989) ("KM") characterize a Nash equilibrium in SFs under uncertainty for an oligopoly; these equilibrium SFs map market price into a level of output. In their model, suppliers bid SFs once into a spot market that is cleared simultaneously with physical production. As noted above in section 1.3, the advantage of supply functions as strategies—as opposed to fixed prices or quantities—is that such functions permit suppliers to adjust output optimally as a function of price in the face of changing or uncertain conditions, for example, uncertainty in demand. KM prove the existence of a Nash equilibrium in SFs for a symmetric oligopoly. If the support of the stochastic demand parameter in this model is unbounded above,[33] there exists a unique, linear SFE.

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[33] And marginal cost and demand are affine for sufficiently large quantity and price, respectively; see Klemperer and Meyer (1989, 1261).

Green and Newbery (1992) and Bolle (1992) were the first authors to apply KM's SFE framework to model (single-settlement) electricity markets. Green and Newbery analyze the British electricity supply industry which, for several years following the 1990 privatization of the Central Electricity Generating Board, primarily comprised two dominant generating firms. They find, like KM, a range of SFEs when the possible variation in demand is bounded. This range is narrowed, however, when they further assume the firms to be capacity-constrained. The authors simulate the British spot market, and include some scenarios that allow for competitive entry. They find, disconcertingly, that even the lower-priced equilibria result in considerable welfare losses. Entry does cause incumbents to bid somewhat lower prices, although the cost in welfare terms of the additional investment is excessive.[34] Bolle (1992) similarly considers SF competition in an electricity spot market, although he does so for a hypothetical market setting. Like the previous authors, he finds a continuum of SF solutions. In contrast to Green and Newbery, Bolle imposes no non-decreasing constraint on the equilibrium SFs which he derives. In some of Bolle's scenarios, the equilibrium SFs are indeed downward-sloping. This suggests that such a non-decreasing constraint—a common feature in real-world electricity markets—may indeed be binding on SFE solutions.[35] A later paper by Bolle (2001) introduces price-sensitive bid functions for both supply- and (some) demand-side market participants competing in a single spot market. He models the remainder of the demand-side entities as non-strategic and having a stochastic level of demand. Bolle finds that, if this non-strategic component of demand

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[34] See Mankiw and Whinston (1986) for a fuller exposition of this phenomenon.

[35] More recently, Baldick and Hogan (2001) have characterized the effect of non-decreasing constraints on SFEs.

is sufficiently large, equilibrium prices may be considerably above marginal cost. In addition, under this condition, market participants might employ mixed strategies.

Rudkevich, Duckworth, and Rosen (1998) develop a useful extension of the single-settlement SFE models of KM and Green and Newbery (1992) discussed above. Namely, the authors relax KM's convexity and differentiability assumptions on firms' marginal cost functions, permitting these to be step functions. For simplicity, the authors consider the case of identical firms. The central analytical result of the investigation is an expression for the market price that results from a symmetric Nash equilibrium in SFs. This price depends on the (stepped) system marginal cost function, instantaneous demand, the maximum demand in the relevant period, and the number of firms. Rudkevich, Duckworth, and Rosen use electricity supply and demand data from Pennsylvania (in 1995) to investigate the properties of this model. The authors compute the average price markups over short-run marginal cost that result from SF bidding in Nash equilibrium. They observe that, while markups do decrease with the number of firms n, electricity prices in the model remain significantly higher than the short-run marginal cost of generation, even for relatively large n. As an example, letting $n = 10$, average markup over marginal cost is still 11%. For fixed n, the authors also investigate how markups vary with (1) the level of capacity non-availability, and (2) the relative error in the day-ahead demand forecast, finding that markups increase monotonically with both of these factors. They conclude that current Commission policies and U.S. antitrust guidelines may not be adequate to mitigate market power in bid-based, competitive electricity markets.

A commonly-cited difficulty in applying SFE models to electricity markets is their computational intractability, particularly when attempting to model transmission network interactions. To overcome this problem, some authors have designed electricity market models that are readily computable. Day, Hobbs, and Pang (2002), for example, introduce a "conjectured supply function (CSF)" approach which, while it resembles an SFE model in some respects, is more closely akin to a general conjectural variations model. A CSF for a given generating firm represents its subjective beliefs concerning the aggregate reaction of its rivals to a change in the market price. Based on a thirteen-bus model of the England and Wales transmission system, the authors find that the CSF approach yields market prices that are "generally more consistent" (p. 8) with those actually observed in England and Wales, compared to the Cournot model. The CSF model, however, is subject to the same criticisms as other conjectural variations models. First among these is an inconsistency between conjectures and firms' actual strategies, absent an explicit requirement of "consistent conjectures" (see, e.g., Bresnahan 1981), which Day, Hobbs, and Pang do not impose. Other shortcomings include restrictive functional form assumptions (the authors use affine CSFs), and arbitrariness in the choice of conjectural parameter (i.e., either the slope or intercept of the affine CSFs) as well as in the conjectured value of the chosen parameter.

Day and Bunn (2001) offer another computational modeling and simulation approach to understanding strategic behavior among SF bidders in electricity markets. Rather than using SFs that are everywhere differentiable, the authors use a grid of discrete price and capacity levels for each competitor over which they define piecewise linear SFs.[36] Since fully flexible SFs of this form produce a nonconvergent cycling of solutions,[37] they impose a bounded rationality constraint on generators' behavior, under which a firm changes the price of only one or two of its plants each day. Day and Bunn apply their methodology to analyze the 1999 generating capacity divestitures in England and Wales. In simulating competition among the three incumbent generating companies and two hypothetical purchasers of varying portions of the incumbents' capacity, the authors find that the increase in the number of competitors from three to five has a marked impact on bid-cost margins. Interestingly, whether incumbents divest 25% or 50% of generating capacity to the two new competitors is of secondary importance in terms of the effect on bid prices. The authors also emphasize the effect of varying degrees of demand elasticity, concluding that at low elasticities of demand (e.g., in the short run), the divestiture of 40% of incumbents' capacity that occurred in 1999 in England and Wales would not mitigate incumbents' market power. Specifically, they find prices in this case in excess of 20% above short-run marginal cost. In the longer run, naturally, they expect higher demand elasticities and market entry to exert downward pressure on these bid-cost margins.

Recently, the California ISO and London Economics International LLC (2003) have developed a comprehensive methodology and computer model for evaluating transmission investments that incorporate strategic SF bidding within a transmission network. The central conceptual problem that the model addresses is the interdependence

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[36] This discretization facilitates the application of an optimization procedure to derive generators' SFs, updating each firm's SF in successive periods to maximize profits based on its rivals' current actions.

[37] Day and Bunn conjecture that this phenomenon is indicative of the existence of mixed-strategy, rather than pure-strategy, Nash equilibria.

of optimal paths for generation and transmission investment (temporally and spatially), given a competitive environment characterized by decentralized, market-based decision making. The main elements of the authors' methodology are simulating imports and exports of power; modeling availability, commitment, and dispatch of hydroelectric and thermal generation; characterizing the entry of new generators over time; and modeling market power. Regarding market power, the authors incorporate two complementary approaches to modeling generators' strategic behavior:

1. A game-theoretic model of strategic bidding (in a discrete strategy space) in which firms conjecture that rivals' current bids are functions of profit-maximizing bids from previous iterations
2. An empirical approach that estimates the historical relationships between data characterizing the state of the market[38] and observed price-cost markups

The authors propose to apply the methodology to evaluate the benefits of a proposed expansion of transmission capacity on "Path 26," the transmission link connecting Southern and Central California.

While the present work is not yet computationally solvable in a network setting, the multi-settlement SFE model compares equilibrium strategies that are *mutually consistent* in all states of the world. That is, in the equilibria we study, a firm's conjectures concerning its rival's strategy coincide precisely with the rival's actual strategy. We do restrict the numerical analysis of chapter 7 to the case of affine spot

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[38] These data include—for each hour and zone—the residual supply index, the total uncommitted capacity of the largest supplier, the system load, and seasonal and zonal dummy variables (California Independent System Operator and LLC 2003).

market demand functions, affine marginal costs, and affine spot market SFs.

1.5.2 Multi-settlement models

Multi-settlement models can capture the potential for firms to take advantage of interactions between the forward and spot markets. The present work proposes an extension of the supply function equilibrium ("SFE") framework developed by KM to a multi-settlement market, whereby demand in each market is uncertain. To the author's knowledge, the present work is the first attempt to use the SF behavioral assumption in a sequential market framework. As intuition would suggest, the resulting SFE in this setting is no longer characterized by a single SF for each supplier, as in KM's model; rather, a sequence of SFs—one in each market—constitutes a supplier's subgame perfect Nash equilibrium (SPNE) strategy in this multi-market framework.[39]

Other authors (e.g., Allaz 1987; Allaz and Vila 1993) consider how the introduction of one or more forward markets, cleared in advance of the spot market, affects competitors' behavior and market outcomes. Of particular interest is the effect on the spot market equilibrium: namely, do forward transactions make the spot market more or less competitive? Allaz and Vila (1993) find that forward market trading is detrimental for firms and beneficial for consumers. Moreover, as the number of forward markets[40] gets large, the outcome in their model approaches the competitive solution. More recently, Ferreira (2003) derives a contrasting result for the case of an infinite number of forward markets. Namely, he finds a set of subgame perfect equilibria that

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[39] Accordingly, unless otherwise specified, the modifier "equilibrium" denotes, throughout this work, the equilibrium concept of subgame perfection (see subsection 3.1.2).

[40] That is, the number of periods in which there is forward trading.

can sustain any outcome between perfect competition and the Cournot outcome.

Looking at how forward markets shape behavioral incentives, Allaz (1987, 18) argues that there are multiple rationales for taking a forward market position. Specifically, he identifies three different motives—*speculative*, *hedging* and *strategic* motives—for forward market participation:

- *Speculative* motives arise from an attempt to profit from price differences between various markets, for example, between a forward market and the spot market. A special case of speculation is pure financial speculation, in which firms do not assume a position in the spot market; instead, they settle any forward obligations financially based on the spot market outcome.
- *Hedging* motives come about from risk aversion in the face of uncertainty. Hedging amounts to purchasing insurance, in other words, accepting a lower expected return in order to achieve a reduction in the variance of returns.
- *Strategic* motives lead market participants to assume forward positions in order to influence the spot market equilibrium.

Allaz (1987, 42 (n. 43)) observes that, under uncertainty, these motives can partly "overlap" in the sense that, for example, "the total position taken [in equilibrium] in the futures market is less than the sum of the strategic and hedging positions if taken separately." Allaz's focus is primarily on strategic considerations, noting that in an oligopolistic setting with perfect foresight (no uncertainty), the strategic motive becomes the only rationale for forward trading. This is because in an equilibrium under perfect foresight, the futures price will be equal to the spot price. Thus, no profits will be made between the forward and spot markets, eliminating any speculative motive. Furthermore, because there is no uncertainty in Allaz' model, there is also no need to hedge.

Allaz (and Vila) rely primarily on an assumption of Cournot conjectures rather than the SFs we focus on in this work.[41] In the present work, we consider SFE competition in a multi-settlement market. We derive a system of ordinary differential equations implicitly characterizing firms' optimal forward market bids, given expectations concerning the spot market. These bids will not, in general, be ex post optimal given the realization of spot market demand. Rather, we will find an *ex post forward market optimum assuming an ex ante expected optimum in the spot market*.[42] We would thus expect that for the multi-settlement market with SFE bidding, strategic motives for forward market participation will be present. In equilibrium, our suppliers also have speculative motives[43] for participating in both the forward and spot markets. Under our assumption that suppliers are risk neutral, however, suppliers have no motive to hedge. Chapter 8 continues this discussion, comparing the results obtained from the present model with Allaz's previous work cited above.

Later work by Newbery (1998) examines a similar—but distinct—sequential market setting of forward (or "contract," in his terminology) and spot markets for electricity in England. His principal findings concern the effect of contracts on entry. Namely, contestable entry and a liquid contract market can enhance efficiency by reducing welfare losses due to the market power of incumbents. Also, capacity

[41] In his 1987 thesis, Allaz does examine several other behavioral assumptions for spot market competition, though not including SFE.

[42] See subsection 3.4.3 for a more precise statement of this notion of optimality.

[43] They are not purely financial speculators, however, since they produce and sell output in the spot market.

constraints tend to attract entry and will increase competition to the extent that new entrants can set prices. Newbery considers general (i.e., nonlinear) SFs and has all demand passing through the spot market; that is, the contracts in his model are purely financial in nature, as are those we study here.

With respect to the present work, it is Newbery's modeling of competition in the forward contract market that is of particular interest. In this market, Newbery has generating firms making "take-it-or-leave-it" offers of a fixed contract quantity at a specified price to consumers. That is, rather than an SF in the forward market, firms offer a *point* in price-quantity space. Newbery analyzes rational expectations equilibria assuming risk-neutral traders, which together imply that the forward contract price is an unbiased estimate of the subsequent spot market price.[44] Newbery acknowledges that more complex contractual forms are possible which could serve to reduce risk for (risk-averse) marketers who have committed themselves to selling at fixed prices and are thus exposed to input price risk. He concludes, however (p. 734, n. 14), that "[l]ittle would be added to the equilibrium-selection story by considering more complex contracts."[45] A more complex contractual environment—that is, contracts based on SF bidding in the forward market—is indeed apposite for modeling multi-settlement electricity markets, although this entails addressing the issue of equilibrium selection to which Newbery alludes. Demand for forward contracts in the present model is uncertain, implying that

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[44] In equilibrium, he finds that the terms offered by suppliers will be such that consumers are indifferent between buying and not buying the contracts; he resolves this knife-edge case in favor of consumers purchasing the offered contract.

[45] Many non-cooperative games—including the one developed here—have multiple equilibria. Equilibrium selection refers to the process of winnowing down the set of these equilibria—perhaps to a unique equilibrium—by invoking plausible (if sometimes ad hoc) criteria such as a Pareto ranking of equilibria, Schelling's (1960) "focal points," stability considerations, etc.

suppliers must submit SF bids in order to respond optimally to this uncertainty. Moreover, the rational expectations assumption—that is, the forward market price equal to the expected spot price—will not hold, absent "sufficient" risk-neutral agents in the model.

Like Newbery (1998), Green (1999a) also examines forward (contract) and spot markets for electricity in England and Wales. In his paper, duopoly electricity generators each choose a quantity of contracts in a forward market while holding a conjectural variation concerning the competitor's forward market response, itself a contract quantity. In the subsequent spot market, each firm bids an SF. Green does not fully motivate his choice of asymmetric behavioral assumptions between the forward and spot markets: the assumption of SFE in the spot market reflects institutional bidding rules for the (now defunct) Electricity Pool, while the assumption of quantity choice in the forward market appears arbitrary. It may be that electricity contracts in England and Wales tend to specify fixed quantities over a wide range of prices, but Green is silent on whether this is so. In the spot market, Green restricts attention to linear SFs, and as in Newbery (1998) above, has all demand passing through the spot market. He considers uncertainty in an appendix to the paper (Green 1999b) in which the intercept of a linear demand function is stochastic when suppliers choose contract quantities, but this uncertainty is resolved before suppliers choose their spot market SFs.

The present framework differs in several important respects from Newbery (1998) and Green (1999a):

1. Here, we assume that firms bid SFs in *both* the forward and spot markets. The assumption of SF bidding in the forward market reflects actual bidding

protocols[46] in centrally-cleared competitive electricity markets, and thus is arguably a more realistic treatment of forward contracting in actual electricity markets. We observe that, given the opportunity in such markets, firms choose to bid an SF rather than a fixed quantity or price.[47] Accordingly, our behavioral model needs, at the least, to accommodate—and indeed, justify—such a choice.

Newbery's concern with equilibrium selection noted above is relevant to the present work as well, as we will also encounter multiple equilibria in the general case studied here.[48] A simplified example (see chapter 5) in which we restrict the analysis to consider only affine spot market SFs has a unique solution in the spot market. In the forward market, a numerical approach to equilibrium selection appears to yield unique optima. We cannot ultimately guarantee, however, that the forward market SFs that we compute constitute globally optimal actions for each firm, rather than merely local optima.

2. There is a demand function in *both* the forward and the spot markets; each of these demand functions, in turn, is subject to exogenous uncertainty as firms submit their SF bids. Forward market demand is endogenous to the expected spot

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[46] These protocols (e.g., California Power Exchange Corporation (2000), Schedule 4, "Bidding and Bid Evaluation Schedule," Section 3.4) commonly specify that participating traders or suppliers must submit a strictly increasing, piecewise linear bid function in the hourly forward energy market. This function gives the quantity of energy that the bidder is willing to supply as a function of the market-clearing price.

[47] In addition, to the extent that *bilateral contracts* have the character of SFs—that is, a contract quantity that increases with price—such contracts would also lend themselves to being modeled via the SFE framework.

[48] Indeed, the problem of multiple equilibria will be aggravated by our assumption of SF bidding in the forward as well as the spot markets. As Newbery (1998, 733) writes, "it would seem natural to model each market as a supply function equilibrium, but not only is there typically a continuum of such equilibria, to each spot market equilibrium there is typically a continuum of contract market equilibria, creating a double infinity of solutions."

market equilibrium and to consumers' private signals concerning the level of spot market demand. Spot market demand also arises endogenously given the technological attributes of consumers and their utility functions.

3. We do not entirely restrict ourselves to affine SFs, as Green does (though we only solve the aforementioned affine example numerically).

More recently, Batstone's (2002) dissertation examines the effects of storage, forward markets, and strategic behavior on competitive electricity markets. The author focuses on characterizing and assessing the risk to which market participants are exposed, and forward contracts' role in hedging this risk. In a two-period model comprising a forward and a spot market, Batstone finds that by exercising market power, strategic suppliers can increase risk for consumers while increasing profits. Similar to the approach in chapter 6 of the present investigation, Batstone derives an endogenous, downward-sloping, forward market demand function, assuming that consumers are risk averse and that they know the distribution of spot market outcomes.[49] The author's "long-run equilibrium" concept[50] (Batstone 2002, subsec. 10.2.2) together with his allowance for "market destabilisation" (ch. 11) amount to a closed-loop[51] (or "feedback") model of the forward and spot markets, an information structure which we invoke in the

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[49] The author distinguishes between the behavior of consumers whose load is unresponsive to price from those whose load is price-sensitive. Given a spot market price distribution, a consumer in the former class maximizes the utility of her total cost of electricity, while a consumer in the latter class maximizes the utility of her net benefit given a spot market demand function. In the present work, we abstract from this distinction among consumers.

[50] This equilibrium provides for a consistency condition to close the model, under which generators and consumers compute distributional moments of spot prices that are consistent with (1) the distribution of hydrological uncertainty and (2) the market equilibrium process of price formation.

[51] See subsection 3.1.1 below for further discussion of the closed-loop concept.

present work, as well. The present investigation is distinct from Batstone's model, however, in some fundamental respects. First, we allow for demand uncertainty in both the forward and spot markets, while Batstone takes demand in each of these markets to be deterministic. The sole source of risk facing suppliers—hydroelectric generators—in Batstone's model is input price risk in the form of marginal water values, which are modeled as stochastic due to uncertain future hydrological conditions. In the present work, in contrast, we assume that cost functions are deterministic. Second, consistent with the assumption of uncertain demand, the present investigation posits competition in SFs in each of the two markets, while Batstone assumes Nash-Cournot conjectures in both markets.

As illustrated by the literature beginning with Green and Newbery (1992) and Bolle (1992), the SFE framework developed here is naturally suited to model bid-based, multi-settlement electricity markets. In the present work, we focus on theoretical foundations, suppressing all but the essential institutional details of actual electricity markets. The central results of this investigation are the derivation and computation of

1. strategic suppliers' optimal bidding strategies and
2. the optimal behavior for a price-taking consumer

within the multi-settlement market setting described above.

1.6 Outline of the thesis

The next chapter, chapter 2, provides a concise overview of the evolution of regulatory policy toward market power in the U.S. electricity sector. Chapter 3 then develops the multi-settlement SFE model. After introducing key concepts used in the model and some notation, this chapter poses the forward market optimization problem for a duopoly supplier bidding SFs in both the forward and spot markets. In chapter 4, we solve this optimization problem analytically for the general case. The sequential nature of the problem suggests backward induction as a solution algorithm. This chapter derives conditions that implicitly characterize the firm's spot and forward market SFs. To obtain an explicit solution for the respective markets' SFs, chapter 5 introduces a number of simplifying assumptions within the model: affine marginal cost and spot market demand functions and affine spot market SF bids. While these simplifications entail some loss of generality, they serve to sharpen the model's results. Next, chapter 6 specifies the characteristics and behavior of consumers. In particular, given risk-averse consumers, it describes how (stochastic) forward and spot market demand functions might arise endogenously. Relying on the simplifying assumptions of chapter 5, chapter 7 derives a singular quasilinear system of ordinary differential equations characterizing the forward market problem and examines the qualitative properties of solutions. For a specific numerical example, this chapter then performs comparative statics analysis with respect to the model's exogenous parameters, and compares welfare results of the multi-settlement SFE model with those of alternative competitive assumptions and market architectures. Based on these results, chapter 8 argues that we might usefully view forward market positions as strategic commitments. It decomposes the motive for forward market activity by suppliers in the multi-settlement SFE model into three distinct effects: a direct effect, a settlement effect, and a strategic effect. This chapter also outlines some extensions that would enhance the model's realism and highlights avenues for further research. Numerous appendices to the thesis collect proofs and other mathematical results.

> The work I have set before me is this . . . how to get rid of the evils of competition while retaining its advantages.
>
> —Alfred Marshall

> We legislate against forestalling and monopoly; we would have a common granary for the poor; but the selfishness which hoards the corn for high prices, is the preventative of famine; and the law of self-preservation is surer policy than any legislation can be.
>
> —Emerson, *Nature: addresses, and lectures*

2 The U.S. policy response to horizontal market power in electricity generation

THIS CHAPTER analyzes how public policy—particularly on the federal level—has responded to horizontal market power as electricity industry restructuring has progressed.[52] Section 2.1 reviews the historical evolution of public policy toward mergers and market-based rates in the electricity industry. Next, section 2.2 focuses on the comparatively recent developments of market power monitoring and mitigation activities, and examines current policies in the various regional markets across the United

[52] Bushnell (2003b); Hieronymus, Henderson, and Berry (2002); and Roach (2002) each provide a useful review and critique of policies to address market power in the various contexts considered in this chapter.

States. Section 2.3 concludes.

2.1 Historical development

As early as Weiss (1975), studies of electricity industry structure and regulation remarked on the potential for the exercise of market power, in the event that regulation of generation were relaxed.[53] In their seminal 1983 book, *Markets for Power: An Analysis of Electric Utility Deregulation*, Paul Joskow and Richard Schmalensee devote an entire chapter (ch. 12) to the subject, examining short- and long-run competition in generation and related antitrust issues. They note (p. 198) that "long-run prospects for market forces to reduce existing levels of concentration seem dim," and, regarding remedies for potential competitive problems in generation, observe that "[e]xisting antitrust rules may not be well-suited to the problems posed by deregulation in this sector; the features of better rules are not apparent. But the need to create better rules before deregulation is clear." These observations foreshadow the extensive conceptual and policy debates on market power and on the appropriate policy responses in the latter half of the 1990s as the restructuring process unfolded.

Historically, public policy toward market power in the U.S. electricity industry took shape in two distinct arenas: (1) regulatory review of utility mergers and acquisitions, and (2) the use of market-based (as opposed to regulated) rates by utilities. We outline below the evolution of policy and the associated analytical methodologies in both of these arenas.

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[53] Weiss notes (p. 165) that "horizontal acquisitions by the largest utilities . . . could have serious anticompetitive effects," although he is cautiously optimistic, on the whole, about potential economies from restructuring and from vertical unbundling, in particular.

2.1.1 Mergers

Pursuant to federal and state antitrust statues, a variety of regulators require firms to demonstrate that proposed mergers or acquisitions would not significantly increase the likelihood of exercise of market power. On the federal level, the Commission assumes the "leading role" in reviewing electric utilities' merger applications; it must approve those that are consistent with the public interest (Pierce 1996, 30). In addition, the Antitrust Division of the U.S. Department of Justice (DOJ) and the U.S. Federal Trade Commission (FTC) may conduct independent reviews to determine whether a proposed merger is consistent with U.S. antitrust laws. Customarily, it has been the Antitrust Division that undertakes such assessments. Rather than conducting its own review, however, the Antitrust Division has in practice limited its activity in the electricity sector to occasional participation in the FERC's investigations (Frankena and Owen 1994, 13). In most states, public utility commissions or state attorneys general review proposed mergers' effects on retail consumers and state utility regulation (Dismukes and Dismukes 1996). Although the domains of institutional responsibility for merger reviews are well-defined by statute and reasonably settled in practice, the associated analytical framework for assessing market power in merger proceedings has evolved over the years along with electricity market architecture and structure (Federal Energy Regulatory Commission 1996f, 1998).

The opinion of the Federal Power Commission (FPC), the FERC's predecessor, in the *Commonwealth Edison Company* case of 1966 (*Commonwealth*)[54] was an early

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[54] Federal Power Commission (1966, 926), *aff'd sub nom. Utility Users League v. FPC*, 394 F.2d 16 (7th Cir. 1968), *cert. denied*, 393 U.S. 953 (1969).

landmark in the application of antitrust principles to electric utilities. In *Commonwealth*, the FPC set forth six criteria that would guide its evaluation of proposed utility mergers:

1. The effect of the proposed action on the applicants' operating costs and rate levels
2. The contemplated accounting treatment
3. Reasonableness of the purchase price
4. Whether the acquiring utility has coerced the to be acquired utility into acceptance of the merger
5. The effect the proposed action may have on the existing competitive situation
6. Whether the consolidation will impair effective regulation either by . . . [the Federal Power] Commission or the appropriate state regulatory authority

For many years afterward, these six so-called "*Commonwealth* criteria" were influential in the FPC's (and, after 1977, the FERC's[55]) treatment of mergers.

The FERC's approach began to change in the 1980s (see Pierce 1996, 31) with the recognition that greater competition in wholesale electricity generation would be both possible and socially desirable.[56] The primary obstacle to such competition was utilities' ability to exclude potential competitors (other utilities and "independent [i.e., non-utility] power producers (IPPs)") from their markets by denying them equal access to their electricity transmission lines. Having no statutory authority to dismantle this competitive

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[55] The FERC was created through the Department of Energy Organization Act on October 1, 1977. It inherited most of the functions of the Federal Power Commission which was eliminated by this Act.

[56] Competitive and regulatory developments in the natural gas industry (in which wholesale sales and interstate pipelines were also under Commission jurisdiction) were further advanced (*Natural Gas Policy Act of 1978* 1978; Federal Energy Regulatory Commission 1985, 1992a; *Natural Gas Wellhead Decontrol Act of 1989* 1989). Increasingly, industry observers cited the accumulating experience and lessons from natural gas as a promising model for electricity industry restructuring (Pierce 1991).

obstacle directly, the Commission resorted to requiring merger applicants to provide open access to their transmission systems under Commission-approved terms ("open access transmission tariffs"). In this way, the Commission began to address vertical constraints to competition by exercising its conditioning authority on a case-by-case basis.[57] These merger proceedings naturally raised horizontal competitive issues, as well, which focused attention on mergers' competitive effects (recall item 5 in the *Commonwealth* criteria above).

In the wake of the *EPAct*, utilities began to undertake mergers and acquisitions at unprecedented rates as they reacted to economic and institutional changes within the industry. It was only in 1996 with its "Inquiry Concerning the Commission's Merger Policy Under the Federal Power Act" and its subsequent "Merger Policy Statement" that the Commission explicitly reconsidered its application of the *Commonwealth* criteria (Federal Energy Regulatory Commission 1996a, 1996f). In its Merger Policy Statement, the Commission asserted that it "will generally take into account three factors in analyzing proposed mergers: the effect on competition, the effect on rates, and the effect on regulation. [Further, the Commission's] analysis of the effect on competition will more precisely identify geographic and product markets and will adopt the Department of

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[57] See Surratt (1998), Moot (1996, 141–42), and Pierce (1996, 30–33) for concise reviews of merger proceedings and the substantive issues involved during the late 1980s and early 1990s. Later, pursuant to the 1992 *EPAct*, the Commission's Order 888 (Federal Energy Regulatory Commission 1996d) required utilities under the Commission's jurisdiction to file open access transmission tariffs.

Justice/Federal Trade Commission Merger Guidelines [DOJ/FTC Guidelines[58]] . . . as the analytical framework for analyzing the effect on competition [p. 3]."[59]

Appendix A of the Merger Policy Statement sets forth the Commission's "Competitive Analysis Screen," which details "a standard analytic method and data specification to allow the Commission to quickly determine whether a proposed merger presents market power concerns."[60] The methodology for evaluating a proposed merger under the Competitive Analysis Screen centers on comparing empirical measures of market concentration[61] with threshold values drawn from the DOJ/FTC Guidelines. The first step in the analysis is to define relevant geographic and product markets and to measure concentration in those markets. The next step is to evaluate post-merger concentration levels and the (pre- to post-merger) change in concentration using the DOJ/FTC Guidelines' concentration thresholds to indicate problematic mergers.

Numerous analysts have taken issue with the Commission's contention that its Merger Policy Statement is consistent with the DOJ/FTC Guidelines. Cox (1999, 28) notes, for example, that the Merger Policy Statement has been criticized for not following the DOJ/FTC Guidelines closely enough, particularly with respect to the method for defining the relevant market. Frankena (1998a, 30–31) goes further, outlining five

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[58] Department of Justice and Federal Trade Commission (1992).

[59] Echoes of *Commonwealth* criteria 1, 5, and 6 from page 43 are apparent in this excerpt from the Commission's Merger Policy Statement.

[60] Federal Energy Regulatory Commission (1996a, App. A, 1).

[61] Like the DOJ/FTC Guidelines, the Commission's Competitive Analysis Screen proposes to measure market concentration by computing the so-called "Herfindahl-Hirschman Index (HHI)" for the relevant geographic and product markets. For any market, the HHI is equal to the sum of firms' squared market shares. The HHI has two appealing properties: (1) it accounts for all firms in a given relevant market, and (2) it gives greater than proportional weight to larger firms' market shares. Stoft (2002, 344) explains the relationship between the HHI and the Cournot competitive model.

respects in which the Commission's Appendix A methodology diverges from the DOJ/FTC Guidelines. Because of these discrepancies, he concludes, the Commission's Appendix A analysis does not constitute a reliable basis for determining the need for antitrust hearings or for fashioning appropriate remedies. Frankena claims elsewhere (1998b, 2), moreover, that "[t]he Appendix A methodology for defining geographic markets leads to substantial violations of the competitive analysis screen standards for some mergers that would not create or enhance market power, and the Appendix A methodology produces no violation for some other potential mergers that would in fact create or enhance market power."[62] Finally, Morris (2000, 176) contends that the Commission's Appendix A methodology appears to overstate a merger's potential anticompetitive effects. Compared with the results of a market simulation model[63] using the same data set, Morris finds that the Appendix A methodology "identifies potential competitive concerns that appear not to exist." The author argues that the discrepancy arises because—unlike the simulation model—the Commission's methodology for market power analysis in the merger context is inherently unrelated to the economic realities of the marketplace.

Moreover, several prominent officials have opined that existing laws and regulations are inadequate to address market power, should it arise in the course of

[62] In particular, Frankena argues (1998b, 2) that the Appendix A analysis could easily produce misleading results with respect to situations involving (1) transmission constraints that limit purchases from multiple sellers and (2) sellers that face opportunity costs. The issue of opportunity costs is central to understanding competition in multi-settlement markets, as explained in subsection 1.4.2 of the present investigation.

[63] Morris uses a standard production cost model for an electricity system: namely, a linear program that computes the production cost-minimizing dispatch to satisfy exogenous demands, with an explicit representation of the transmission network's physical properties included in the constraint set.

electricity restructuring. For example, Joel Klein, a recent head of the DOJ's Antitrust Division (under the Clinton Administration), noted that "[t]he antitrust laws provide ample authority for the Justice Department to challenge anticompetitive conduct of various sorts, but we cannot challenge market structure itself. In other words, to whatever extent restructured electric power markets are too highly concentrated to yield pricing at or near competitive levels, the antitrust laws provide no remedy" (Klein 1998). Klein's deputy, A. Douglas Melamed, later observed that "[t]he antitrust laws do not outlaw the mere possession of monopoly power that is the result of skill, accident, or a previous regulatory regime. Antitrust remedies are thus not well-suited to address problems of market power in the electric power industry that result from existing high levels of concentration in generation or vertical integration" (Melamed 1999).[64] Thus, apart from remedying any shortcomings in analytical methodology, the Commission may require new enforcement authority to mitigate some instances of electricity sector market power outside the context of merger reviews. Failure to create such authority may jeopardize the efficiency gains from electricity sector restructuring while creating oligopoly rents to suppliers with market power.

The comprehensive energy bill introduced in the Senate in February 2004 (Senate 2004) would reform and clarify the Commission's merger authority in several ways, but does not provide explicit guidance on the conduct of market power analysis in merger cases. First, the bill raises the monetary thresholds for mergers and acquisitions to be

[64] Melamed's words echo the U.S. Supreme Court's opinion in a seminal antitrust case, *United States v. Grinnell Corp.* (384 U.S. 563 (1966)), which established that market power ("monopoly power," in the Court's language) attained only "from growth or development as a consequence of a superior product, business acumen, or historical accident" is not objectionable under the U.S. antitrust statutes.

subject to Commission review. Next, in evaluating whether a merger or acquisition is in the public interest, the Commission is to consider adequate protection of consumer interests, consistency of the transaction with competitive wholesale markets, and the effects on the financial integrity of the transacting parties (among other criteria that the Commission may deem consistent with the public interest). In addition, the Commission is to develop procedures for expedited review of mergers and acquisitions, identifying classes of these transactions that normally meet these public interest standards. The Commission is required to report annually to Congress any conditions imposed in the preceding year on utility mergers and acquisitions, and justify these under a public interest standard. Finally, the Secretary of Energy is charged with studying the extent to which the Commission's authority under section 203 of the *Federal Power Act* to review utility mergers and acquisitions is duplicated elsewhere, and with making recommendations to eliminate any unnecessary duplication or delays in such reviews.

2.1.2 Market-based rates

A combination of deregulatory legislation[65] and technical advances in natural gas-fired generating technologies facilitated the growth of a competitive threat to incumbent utilities' customer base from IPPs and "exempt wholesale generators (EWGs)."[66] While the *EPAct* provided only that IPPs and EWGs could sell *wholesale* power to utilities, these producers along with large consumers (e.g., industrial plants) had natural incentives

[65] In particular, the *Public Utility Regulatory Policies Act of 1978*, the *Natural Gas Policy Act of 1978*, the *Natural Gas Wellhead Decontrol Act of 1989*, and the *EPAct*.

[66] Surratt (1998, 24). Under the *EPAct* (*U.S. Code*, vol. 15, sec. 79z-5a), an exempt wholesale generator denotes an electric power producer (a utility affiliate or an independent) that sells electricity at wholesale and that the Commission has exempted from the provisions of the *Public Utility Holding Company Act of 1935*.

to pursue direct *retail* sales arrangements with each other. Known as "retail wheeling," realizing these transactions usually required access to utilities' transmission lines. Under the *EPAct*, the right to compel utilities to provide such (retail) access was reserved to the states. By mid-1993—less than one year after passage of the *EPAct*—at least eight states had legislative or regulatory proceedings underway examining the merits of retail wheeling (Anderson 1993, 16–18).

One avenue that utilities pursued to meet this competitive threat was to seek authority from the Commission to use *market-based rates* (i.e., unregulated rates) for wholesale power sales. Market-based rates give utilities flexibility with respect to rate levels and structure, which would be essential in retaining customers that were able, increasingly, to choose their electricity supplier. As it noted in the *Ocean State Power* case (Federal Energy Regulatory Commission 1988, 61979), the Commission has discretion to depart from cost-based ratemaking "when necessary or appropriate to serve a legitimate statutory objective of the *Federal Power Act*." *Ocean State Power* also documents the historical evolution of the Commission's market-based rate policy and outlines in general terms the Commission's threshold test for permitting market-based rates (or, as characterized below, "market-oriented pricing"): "Generally, the Commission can rely on market-oriented pricing for determining whether a rate is just and reasonable when a workably competitive market exists, . . . or when the seller does not possess significant market power. . . . A seller lacks significant market power if the seller is unable to increase prices by restricting supply or by denying the customer access to alternative sellers. Lack of market power is the key prerequisite for allowing market-oriented pricing" (p. 61979 (references omitted)).

The *Doswell Limited Partnership* proceeding (Federal Energy Regulatory Commission 1990) ("*Doswell*") helped to define further the substance of the Commission's market power test for market-based rate cases. The background of *Doswell* was a competitive solicitation of bids for electrical generating capacity in 1987 by Virginia Electric and Power Company ("Virginia Power"). Based on the solicitation, Virginia Power agreed to purchase capacity from the Intercontinental Energy Corporation ("Intercontinental"), among other suppliers. Intercontinental later assigned its purchase agreements to the Doswell Limited Partnership ("Doswell"), and Doswell filed the market-based rates proposed in these agreements in late 1989 with the Commission. In its *Doswell* Order, the Commission held that

> [t]here are several factors that lead us to conclude that both Intercontinental and its successor, Doswell, lacked market power over Virginia Power. First, Intercontinental did not own or control, and was not affiliated with any entity that owned or controlled, transmission facilities within or around the Virginia Power service area, other than those necessary to interconnect with Virginia Power for this sale. Therefore, Intercontinental was not in a position to prevent Virginia Power from reaching competing suppliers. . . . Second, there is no evidence that Intercontinental or Doswell was a dominant firm in any generating market that might be relevant to providing capacity and energy to Virginia Power. . . . Third, there is no evidence that either Intercontinental or Doswell controlled resources that allowed it to erect any other barrier to potential competing generation suppliers (Federal Energy Regulatory Commission 1990, 61757–58).

Throughout the 1990s[67]—indeed, until the Order in Federal Energy Regulatory Commission (2001c), the Commission would grant market-based rates to an applicant "if the seller [i.e., the applicant] and its affiliates do not have, or have adequately mitigated, market power in generation and transmission and cannot erect other barriers to entry" (p. 61969), echoing the structure of the Commission's market power test in *Doswell*. Only the second component of the market power test, that for generation market power, has

[67] Surratt (1998, 24–27) and Raskin (1998b, 17–18) trace the evolution of the Commission's market power analysis through its various decisions in market-based rate proceedings focusing, in particular, on generation market power.

required that the applicant perform an analytical test.[68] This analytical test for generation market power has come to be known as the "hub-and-spoke" test.

The hub-and-spoke test begins—as do the DOJ/FTC Guidelines, discussed above—by defining relevant geographic and product markets. As Bohn, Celebi, and Hanser explain (2002, 53–54), "The hub-and-spoke test defines relevant geographic market as the combination of applicant's destination market (the hub) plus the set of all markets that are directly connected to the destination market (the spokes).[[69]] Product markets are generally defined as installed and uncommitted capacity. The test involves a comparison of the share of generation resources controlled by the applicant and its affiliates to that of all owners of generation within the relevant geographic markets. . . . The Commission has generally interpreted a market share of less than 20% as evidence of a lack of horizontal market power."

The Commission's own Merger Policy Statement (Federal Energy Regulatory Commission 1996f, 20–21) described the shortcomings of the hub-and-spoke analysis:

> [The hub-and-spoke method] defines geographic markets in a manner that does not always reflect accurately the economic and physical ability of potential suppliers to access buyers in the market. . . .
>
> . . . [I]t does not account for the range of parameters that affect the scope of trade: relative generation prices, transmission prices, losses, and transmission constraints. Taking these factors into account, markets could be broader or narrower than the first- or second-tier entities identified under the hub-and-spoke analysis. . . . In other words, mere proximity is not always indicative of whether a supplier is an economic alternative.

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[68] As for the other two components of the market power test, if the market-based rate applicant and its affiliates have filed an open access transmission tariff with the Commission, this has been sufficient to demonstrate the absence (or mitigation) of transmission market power. Regarding barriers to entry, the Commission "relies on an applicant's representation and public policing" (Federal Energy Regulatory Commission 2001c, 61969).

[69] Generation market power analyses sometimes refer to the suppliers connected to the destination market by these "spokes" as "first-tier" (or "tier one") suppliers. "Second-tier" (or "tier two") suppliers are those suppliers directly interconnected with the applicant and which the customer in the destination market can reach via the applicant's open access transmission tariff. See Federal Energy Regulatory Commission (1992b, 61757) and Dalton (1997, 35).

The Commission did not elaborate its reasoning why, despite the deficiencies that it noted in the merger context, the hub-and-spoke analysis continued to be suitable for market-based rate cases.

In a concurring opinion some years later, FERC Commissioner William L. Massey offered his perspective on the deficiencies of the hub-and-spoke analysis (Federal Energy Regulatory Commission 2000, 2):

> I have come to believe that [hub-and-spoke analysis] is an anachronism. This method focuses solely on the market share of the individual seller instead of the conditions in the market. It assumes that all sellers that are directly interconnected with the customer, and all sellers directly interconnected with the applicant for market-based rates, can reach the market, and market shares are evaluated on that basis.
>
> This is a back of the envelope approach, more or less. It takes little or no account of the important factors that determine the scope of electricity markets, such as physical limitations on market size including transmission constraints, prices, costs, transmission rates, and the variance of supply and demand over time. The hub and spoke is much too primitive for these times. Clearly, the Commission must develop a more sophisticated approach to market analysis, and I would recommend that we proceed generically to do so.

Speaking to the Energy Bar Association one year later (Massey 2001, 6), Massey's impatience with the Commission was palpable: "Any market participant that cannot pass [the hub-and-spoke] test needs a new lawyer. How accurate can this test be? How much faith can state commissioners have in our market based pricing policy if we still use this horse and buggy analytical approach? Relying on the hub and spoke is sheer folly."

Industry analysts outside of the Commission have also weighed in regarding the flaws in the hub-and-spoke approach. For example, Bohn, Celebi, and Hanser (2002, 54) share Massey's misgivings. They note that the Commission has generally construed a market share of less than 20% in the hub-and-spoke test as a lack of evidence of market power, though this figure has not served as a "bright line" standard. The authors argue that this threshold concentration level is fundamentally arbitrary. Critically, it fails to identify electricity suppliers having lower market shares that, when markets are "tight," may be able to exercise market power. Perhaps the most detailed and vociferous critique of the hub-and-spoke approach is Stoft (2001, 1). He demonstrates that the hub-and-spoke test is flawed in the following respects:

- Its geographical market definition accounts only for a factor that is no longer relevant and for none of the factors that matter in a competitive market.
- Its use of uncommitted-capacity shares registers more market power when the market itself is more competitive and less market power when it is less competitive. Thus it often reads in reverse the impact of the market on the applicant.
- It takes no account of the central market-power problem of electricity markets: the inelasticity of demand.
- It takes no account of the thousand-fold fluctuations in supply elasticity that concentrate and intensify market power during a few crucial hours.
- It takes no account of suppliers becoming pivotal to the market.
- It would allow a single supplier to pass its screen although it possessed enough market power to single-handedly double the average year-round price in a market as well behaved as PJM's.
- It would allow multiple suppliers to pass although they would be capable of destroying any current power market.

Stoft concludes that "[s]uch a 'screen' misinforms, serves no useful purpose and should be immediately discontinued" (p. 1).

Responding to the growing dissatisfaction with and criticism of its hub-and-spoke analysis of generation market power in market-based rate cases, the Commission concluded in a November 2001 order (Federal Energy Regulatory Commission 2001c, 61969) that "because of significant structural changes and corporate realignments that have occurred and continue to occur in the electric industry, our hub-and-spoke analysis no longer adequately protects customers against generation market power in all circumstances. The hub-and-spoke analysis worked reasonably well for almost a decade when the markets were essentially vertical monopolies trading on the margin and retail loads were only partially exposed to the market." This order also introduced a new analytical screen—the "Supply Margin Assessment (SMA)"—to replace the hub-and-spoke analysis. In essence, the SMA screen evaluates whether a market-based rate applicant is "pivotal" in the market, that is, whether at least some of the applicant's capacity is needed to satisfy the market's peak demand. If an applicant is deemed pivotal, it does not pass the screen and its spot market energy sales will be priced using cost-based rates; moreover, the applicant must publicize projected incremental cost data to help buyers make rational purchasing decisions. The SMA screen applies to market-based rate applications and triennial reviews of market-based pricing authority on an interim basis pending a re-examination of the Commission's methods of market power analysis.[70] Sales of energy in FERC-approved ISOs or RTOs, however, are exempt from the SMA screen.

While generally acknowledging its improvements over the hub-and-spoke test, a few authors have called attention to potential drawbacks of the SMA screen. Rohrbach,

[70] According to the Commission (Federal Energy Regulatory Commission 2001c, 61969), the SMA screen improves upon the Commission's former hub-and-spoke analysis in two respects. First, the SMA screen considers the effect of transmission constraints on geographic market definition. Second, the screen establishes a threshold based on whether a firm is pivotal in its market.

Kleit, and Nelson (2002), for example, contend that "the SMA [screen] does not adequately resolve a number of critical issues and raises new ones" (p. 11). They observe that the SMA screen does not require that the potential exercise of market power be profitable (p. 12); hence, firms that would not profit if they were to exercise market power would still not pass the SMA screen. Bohn, Celebi, and Hanser (2002, 54) note that the SMA screen improves on the hub-and-spoke test by modeling relevant transmission constraints via "total transfer capability (TTC)."[71] Nonetheless, the use of TTC has its own drawbacks: the deficiencies of TTC and related metrics based on "transfer capability"—due to the reality of loop flow in the transmission system and its associated economic effects—are by now well-known (see, e.g., Hogan 1992, 215–16; Harvey, Hogan and Pope 1997, 8–21). Bohn, Celebi, and Hanser suggest several refinements to the SMA test to account for factors that it currently ignores, including the following: diurnal and seasonal demand variations, import capability when the applicant controls capacity outside of the market under study, simultaneous import limits (which are not accounted for by TTC), collusive exercise of market power, derating installed capacity for unit outages, flexibility in generating plant operations (i.e., distinguishing plants that may readily vary their output from inflexible—e.g., nuclear—plants), retail

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[71] The North American Electric Reliability Council(NERC)'s (1996) definition of total transfer capability (TTC) is, in essence, as follows:

> The amount of electric power that can be transferred over the interconnected transmission network in a *reliable* manner based on . . . the following conditions:
>
> 1. For the existing or planned system configuration, and with normal (precontingency) operating procedures in effect, all facility loadings are within normal ratings and all voltages are within normal limits.
> 2. The electric systems are capable of absorbing the dynamic power swings, and remaining stable, following a disturbance that results in the loss of any single electric system element, such as a transmission line, transformer, or generating unit.
>
> . . . [See the cited source for additional details].

load obligations, and compatibility with the Commission's Appendix A methodology applied in merger proceedings.

To provide a venue for discussion of the merits of the SMA screen, the Commission convened a Technical Conference in January 2004 (Federal Energy Regulatory Commission 2004f). The Conference's agenda included geographic market definition, accounting for transmission limitations, the appropriate interim screen for generation dominance, and appropriate mitigation measures for utilities that fail the generation dominance screen.

A companion FERC Order to the November 2001 SMA Order proposes revising existing market-based rate tariffs by explicitly proscribing anticompetitive behavior and the exercise of market power (Federal Energy Regulatory Commission 2001b, 1). The proposed tariff provision is as follows: "As a condition of obtaining and retaining market-based rate authority, the seller is prohibited from engaging in anticompetitive behavior or the exercise of market power" (p. 4). The Order continues, defining these terms: "Anticompetitive behavior or exercises of market power include behavior that raises the market price through physical or economic withholding of supplies. Such behavior may involve an individual supplier withholding supplies, or a group of suppliers jointly colluding to do so. Physical withholding occurs when a supplier fails to offer its output to the market during periods when the market price exceeds the supplier's full incremental costs. . . . Economic withholding occurs when a supplier offers output to the market at a price that is above both its full incremental costs and the market price (and thus, the output is not sold)" (p. 4).[72]

Various commenters criticized the above tariff provision for vague definitions of economic and physical withholding,[73] arguing that "full incremental costs," in the Commission's parlance, would need to account for opportunity costs due to multiple markets across time, space, and various products (e.g., energy vs. generation reserves), and voiced fears that this new measure would create increased regulatory risk, deterring needed investment and entry in the industry. Informed by intervenors' comments, behavior observed in the Western markets of the United States (see, e.g., Federal Energy Regulatory Commission 2003a), accumulating experience with other U.S. electricity markets (particularly in the East), and FERC public conferences, the Commission issued an Order in November 2003 (Federal Energy Regulatory Commission 2003b) ("MBR Tariff Order") conditioning new and existing market-based rate tariffs on sellers' compliance with six "Market Behavior Rules,"[74] summarized below:

1. Generation unit scheduling, bidding, operation, and maintenance in compliance with Commission-approved rules and regulations
2. Prohibition on market manipulation, that is, transactions without a legitimate business purpose that are intended to or foreseeably could manipulate market

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[72] In response to several procedural motions shortly after this order, however, the Commission deferred the effective date of the proposed tariff provision (Federal Energy Regulatory Commission 2001a) and granted rehearing of the order for further consideration (Federal Energy Regulatory Commission 2002b).

[73] A common assertion made by commenters was that, due to various technical and institutional features of the industry, price-taking behavior would likely be mis-classified under the tariff provision as economic and physical withholding, thus inviting the charge that the firm in question had exercised market power.

[74] Appendix A to Federal Energy Regulatory Commission (2003b), pp. 65–66.

prices, conditions, or rules

3. Provision of accurate, factual information in communication with the Commission, market monitors, RTOs, ISOs, and transmission providers
4. Accurate and factual reporting of information to publishers of electricity or natural gas price indices (to the extent that a seller engages in such reporting)
5. Retention of data and information that explains prices charged for electric energy and related products for a three-year period
6. No violation or collusion with another party in violation of a seller's market-based rate code of conduct

The Commission received numerous requests for rehearing of its MBR Tariff Order, and in January 2004, it granted rehearing of this Order for further consideration (Federal Energy Regulatory Commission 2004d).

2.1.3 Discussion

Bushnell (2003b, 12) has noted that typically, regulatory decisions to grant market-based rate authority had a greater impact on the progress of electricity restructuring in the United States than did merger approvals. In the former instance, many entities applying for market-based rates—power marketers, for example—were and are not subject to state-level regulation of retail sales. Granting authority to these market participants to charge market-based rates for wholesale sales amounted to the removal of the only constraint on such firms' pricing behavior. As for mergers, these have been between regulated utilities, for the most part, so that both merging parties as well as the new post-merger entity are subject to retail rate regulation.

This is not to say that merger approvals are inconsequential as a matter of policy.

Undoing a utility merger once it has been consummated would likely be simply infeasible. On the other hand, revoking a utility's market-based rate authority would be a relatively straightforward matter, entailing only an administrative order. Pursuant to Federal Energy Regulatory Commission (1996c), sellers with market-based rate authority are required every three years to update the market power analysis underlying the grant of such authority.

2.2 Market power monitoring and mitigation

2.2.1 Origins

In response to the California Public Utilities Commission's restructuring order in December 1995,[75] California's three investor-owned utilities (IOUs) filed applications with the FERC for market-based pricing authority. Citing the FERC's growing concern with the implications of transmission constraints for geographic market definition in the context of market power analysis, the three utilities proposed in their joint filing[76] to account for such constraints in their (forthcoming) market power analyses. In the event, one of the three California IOUs, Pacific Gas & Electric (PG&E) Co., submitted a separate market power analysis, while the other two, Southern California Edison (SCE) and San Diego Gas and Electric (SDG&E), conducted a joint study.[77] Significantly, apart from market power analysis and some recommended market power mitigation measures,

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[75] California Public Utilities Commission (1995), as corrected by California Public Utilities Commission (1996).

[76] Pacific Gas and Electric Company (1996). Federal Energy Regulatory Commission (1996b) elaborates the Commission's concerns regarding transmission constraints.

[77] These analyses are Pacific Gas & Electric Co. (1996), and Southern California Edison and San Diego Gas and Electric (1996).

these two filings each contained a proposed "monitoring" program for market power. PG&E's proposal (1996, 24) recommended that a monitoring program be "administered and run by a Compliance Division of the [California] PX, similar to the compliance divisions that exist within the stock exchanges, as well as the New York Mercantile Exchange."[78] Similarly, SCE and SDG&E recommended that the Commission require that "[a] three-year monitoring program, administered by the [California] ISO, be put in place at the time the PX begins operating. The monitoring program would be designed to collect information on market behavior and performance that the Commission could rely upon to evaluate complaints, analyze proposals to fine tune operating details, and come to a final conclusion that the market's performance meets the Commission's standards for just and reasonable rates" (1996).

In its December 18 order (Federal Energy Regulatory Commission 1996e, 27–28), the Commission required that the three utilities file additional information on their market power mitigation plans, agreed with SCE and SDG&E's earlier suggestion (1996, transmittal letter 6–7) to convene a technical conference on market power mitigation options, and directed the California ISO to file a detailed "monitoring plan," addressing the following considerations:

- Who is responsible for the monitoring;
- What information would be collected;
- What the criteria for identifying the exercise of market power would be;
- What reports and information would be submitted to the Commission; and

[78] The California Power Exchange (PX) was an independent, non-profit entity designed to manage the forward energy markets in California in conjunction with the ISO. The PX suspended operation of its day-ahead and day-of markets on January 31, 2001 and filed for bankruptcy protection on March 9, 2001.

- What mitigation actions would be taken if the exercise of market power is identified.

With this policy decision, the function of "market monitoring" was born.[79] One industry observer saw two primary motivations underlying the Commission's charge to the California ISO to institutionalize a market monitoring capacity in its emerging competitive market (Lock 1998a). First, early deregulatory reforms in Chile and in England and Wales notwithstanding, the Commission recognized how little experience had been gained, to date, with the proposed auction-based markets. Second, the Commission was cognizant that—as argued in subsection 2.1.1 above—the antitrust agencies lacked the statutory authority to address many market power concerns, while the Commission itself did not have the technical capacity to perform effective monitoring. Another analyst has argued that the Commission's order of December 18, 1996 (Federal Energy Regulatory Commission 1996e) signaled a significant change in the Commission's policy toward market power, in that the Commission "intended to shift its focus from an analysis of market structure to reliance on mitigation measures to ensure that generation owners would not exercise market power" (Raskin 1998a). In the years following that order, the Commission imposed a similar market monitoring requirement for the three ISOs in the northeastern United States—ISO-NE, NYISO, and PJM—as they developed their market architectures.

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[79] While this represented the first incarnation of market monitoring in the United States, Lock (1998b, 18) notes that Alberta, Canada required as part of the *Alberta Electric Utilities Act* (*Alberta Electric Utilities Act of 1995* 1995, section 9(1)(d)) that "[t]he Power Pool Council [of Alberta] shall . . . monitor the performance of the power pool and change the rules of the power pool, if necessary, to promote an efficient, fair and openly competitive market for electricity." The inception of and early experience with Alberta's "market surveillance" system is discussed in Barker, Tenenbaum, and Woolf (1997, 40–45).

The Commission placed market monitoring on a more secure institutional footing with Order 2000 on Regional Transmission Organizations (1999) which proposed, among many other provisions regarding management of the transmission grid, that RTOs perform market monitoring as one of their "core functions":[80] "Specifically, RTOs would be required to: (1) monitor markets for transmission service and the behavior of transmission owners and propose appropriate action; (2) monitor ancillary services and bulk power markets that the RTO operates; (3) periodically assess how behavior in markets operated by others affects RTO operations and how RTO operations affect those markets; and (4) provide reports on market power abuses and market design flaws to the Commission and affected regulatory authorities, including specific recommendations (Federal Energy Regulatory Commission 1999, 435)." Each of the five FERC-jurisdictional U.S. ISOs (see note 2) created a specialized entity to perform the market monitoring function[81] as the ISOs—along with all FERC-jurisdictional public utilities—undertook to comply with Order 2000.

The authority and responsibilities of these market monitoring organizations are similar—though not identical—across the ISOs. In general, ISO tariffs empower the market monitoring organizations to perform the following tasks:

- The objective of market monitoring is to identify any exercises of market power, abuse of market rules, or market design flaws. To this end, monitoring organizations collect data on the operation of all product markets (e.g., energy,

[80] Whereby either the RTO itself or an independent entity created by or under contract with the RTO may carry out the monitoring activities.

[81] We refer to these entities generically as "market monitoring organizations" and identify the particular organizations within the various ISOs in the next subsection below.

reserves of various kinds, capacity) administered by the respective ISOs—and in some cases, bilateral markets—on an ongoing basis. Market competitiveness and economic efficiency are the overarching standards of interest to monitoring organizations, for which they have developed a variety of indicators (see PJM Interconnection 2001 for an example of a comprehensive list). While specific methodologies and analytical procedures vary among the ISOs, common indicators include assessments of generation ownership concentration—where the "relevant" market accounts for transmission constraints—using HHIs (see note 61), comparisons of bids and market prices to unit-specific cost data (accounting for the prices of fuel and other inputs, and sometimes using cost-based dispatch simulation models), changes in bidding behavior over time, and declarations of generation unit availability.

- Take corrective actions, for example, some monitoring organizations can make price corrections resulting from software or data entry errors.
- Recommend changes in market rules or in market monitoring procedures to the governing board or stakeholder committee which, if approved, are then filed with the Commission for regulatory approval.
- Assist the Commission or antitrust enforcement agencies in investigations that they may undertake.

With its Standard Market Design Notice of Proposed Rulemaking ("SMD NOPR") (Federal Energy Regulatory Commission 2002a), the Commission proposed three mandatory market power mitigation measures and one such voluntary measure as components of all jurisdictional utilities' (and RTOs') open-access transmission tariffs (p.

222). The first measure targets local market power possessed, in particular, by generating units that must run to support reliability of the transmission network. At times when such units have market power, their bids should be capped. The second monitoring provision of the SMD NOPR is a "safety-net" bid cap of $1,000/MWh to apply at all times and locations, serving as a check on the degree of generators' economic withholding. Third, the SMD NOPR envisions a resource adequacy requirement on a regional basis to ensure reliability. This requirement does not address withholding directly; rather, it is designed to diminish "the ability and incentive of suppliers to practice and profit from either physical or economic withholding" (p. 223). The fourth, voluntary, measure is intended to apply at times when non-competitive conditions exist. Market operators would examine suppliers' bids and, if withholding—rather than scarcity—is responsible for the level of such bids, possibly mitigate these bids. Certain predetermined conditions or triggers, or infrastructural constraints[82] could prompt the imposition of such a mitigation measure. Responding to extensive comments on its SMD NOPR, the Commission issued in April 2003 a White Paper on a "Wholesale Power Market Platform" (Federal Energy Regulatory Commission 2003d) outlining its vision for further electricity industry restructuring and sketching proposed changes to the SMD NOPR. In this White Paper, the Commission emphasized the fundamental balance that market power mitigation measures must strike, namely, to "protect against the exercise of market power without suppressing prices below the level necessary to attract needed investment in new infrastructure . . ." (Federal Energy Regulatory Commission 2003d, 8). Specifically,

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[82] Such as drought in a system relying significantly on hydropower resources (p. 223).

RTO tariffs would be required, at a minimum, to limit bidding flexibility in the presence of local market power and to prevent market manipulation strategies (p. 8).

Together with the monitoring and mitigation provisions of Standard Market Design noted above, a recent institutional innovation at the Commission may encourage the development and application of a coherent analytical framework for curbing market power. Namely, in January 2002, the Commission created the Office of Market Oversight and Investigation (OMOI), a new monitoring unit at the federal level for energy markets. The OMOI has as its mission to "to protect customers through understanding markets and their regulation, identifying and fixing market problems, and assuring compliance with Commission rules and regulations" (Federal Energy Regulatory Commission 2004b). Among the OMOI's functions are (Federal Energy Regulatory Commission 2004a):

- Undertaking market research, modeling, and simulation; maintaining data resources in support of oversight and investigatory activities
- Conducting analyses of energy markets, providing early warning of vulnerable market conditions, and proposing appropriate policies
- Investigating possible violations of Commission rules and regulations, recommending remedies to address violations and, where authorized, pursuing these remedies
- Maintaining a forum for resolving disputes informally and advising the Commission on questions of enforcement and compliance

2.2.2 Monitoring and mitigation in regional markets

ISO staffs and budgets devoted to market power monitoring and mitigation have grown markedly over time as the extent and complexity of monitoring has increased (Peterson et al. 2001, 20). Table 2.1 below provides an overview of market power monitoring and mitigation organizations and the protocols or plans that they implement in each of the six ISOs in the United States.[83]

[83] See also Goldman, Lesieutre, and Bartholomew (2004), Kinzelman (2002), Power Pool of Alberta (2002, 42), Peterson et al. (2001), and Energy Regulators Regional Association (2001) for a more detailed discussion and comparisons of individual ISOs' monitoring activities and experiences.

TABLE 2.1: OVERVIEW OF ISO MARKET POWER MONITORING AND MITIGATION PLANS

	Date that operations commenced	*Monitoring organizations*		*Monitoring protocol(s)*
		Internal	*External*	
CAISO	March 31, 1998	Market Surveillance Unit (MSU) (1998–1999), Department of Market Analysis (DMA) (1999–)	Market Surveillance Committee (MSC)[a] Electricity Oversight Board (EOB)[b]	ISO Market Monitoring & Information Protocol (MMIP) (California Independent System Operator 2003b); ISO Enforcement Protocol (proposed)[f] (Att. B of California Independent System Operator 2003a)
ERCOT	July 31, 2001	None	Public Utility Commission of Texas (PUCT) Market Oversight Division	ERCOT Protocols, Section 17: Market Data Collection and Use (ERCOT 2001); Order Adopting New §§25.90, 25.91 and 25.401 (Public Utility Commission of Texas 2000)
ISO-NE	May 1, 1999	Market Monitoring and Market Power Mitigation Section	Independent Market Advisor (IMA)[c]	New England Power Pool and ISO New England, Inc. (2003)
MISO	February 1, 2002	None	Independent Market Monitor (IMM)[d]	Module D: Market Monitoring and Mitigation Measures (proposed) (Midwest ISO 2004);[g] Attachment S-1: Independent Market Monitor Retention Agreement (Midwest ISO 2002b)
NYISO	December 1, 1999	Market Monitoring Unit (MMU)	Market Advisor[e]	ISO Market Power Mitigation Measures (New York Independent System Operator 2004b)[h]
PJM	April 1, 1997	Market Monitoring Unit (MMU)	None	PJM Market Monitoring Plan (PJM Interconnection 2003)

Notes:

[a] The MSC is an external, independent market advisory body consisting of three (later increased to four) experts in antitrust economics and industrial organization as well as utility law, regulation, and operations.

[b] California's Electricity Oversight Board (EOB) was established by Chapter 854, Statutes of 1996 (AB 1890), comprising state legislators and appointees of California's governor. The EOB's initial task was to select the Boards of Directors for the CAISO and PX. In addition, the EOB oversees the activities of the CAISO, and conducts analysis and drafts recommendations regarding market operation, system reliability, and infrastructure planning.

[c] The ISO-NE Board of Directors retains an Independent Market Advisor (IMA) to provide market analysis and advice directly to the Board on making the ISO-NE markets more competitive and efficient. ISO New England ([n.d.]) describes the circumstances in which the ISO's Market Monitoring and Market Power Mitigation Section typically uses the services of the IMA.

Notes to Table 2.1 (cont'd):

[d] The MISO's Independent Market Monitor (IMM) has "experience and expertise appropriate to the analysis of competitive conditions in markets for energy, ancillary services, and transmission rights. . . ." The IMM advises the MISO and reports to the Commission regarding "the nature and extent of, and any impediments to, competition in and the economic efficiency of the Midwest ISO's Markets and Services; . . . " (Midwest ISO 2002a, secs. 4.1 and 4.3).

[e] The Market Advisor in the NYISO has "experience and expertise appropriate to the analysis of competitive conditions in markets for electric capacity, energy and ancillary services, and financial instruments such as TCCs. . . ." The Market Advisor reports to the NYISO Board of Directors on the "nature and extent of, and any impediments to, competition in and the economic efficiency of the New York electric Markets . . ." (New York Independent System Operator 1999, secs. 4.1 and 4.3).

[f] In Federal Energy Regulatory Commission (2004e), the Commission ordered the CAISO to modify this proposed Protocol. The CAISO objected to the required modifications, however, and requested rehearing and clarification of the Commission's Order (California Independent System Operator 2004).

[g] Filed on March 31, 2004 with effective date of December 1, 2004, pending Commission approval.

[h] This version of the Market Power Mitigation Measures for the NYISO was submitted as part of a compliance filing to the Commission (New York Independent System Operator 2004a), pursuant to Federal Energy Regulatory Commission (2004c). It has an effective date of May 1, 2004, contingent on its acceptance by the Commission.

2.3 Assessment

The evolving standards for merger and market-based rate cases reviewed in subsection 2.1 suggest not only a dynamic electricity industry, but also a lack of consensus—both within the industry or between the industry and the Commission—regarding appropriate criteria and methodology for market power analysis. At this writing, methods for market power assessment in market-based rate proceedings remain subject to rehearing some two-and-a-half years after the Commission first proposed revising these methods (Federal Energy Regulatory Commission 2004d). Commissioner Brownell's concurring opinion in Federal Energy Regulatory Commission (2003c) emphasized that basic theoretical questions persist in this regard: "I . . . have a fundamental concern that we've allowed markets to form without a full appreciation of what constitutes a market let alone the market dynamics that foster a truly competitive market. For example, what defines a competitive market and what constitutes scarcity pricing? These questions remain largely unanswered (p. 26)." The SMD proceeding has arguably sharpened the focus of the debate on market power and a host of other market design issues while also highlighting the significance of regional differences in economic structure, market development, and timing of reforms (Federal Energy Regulatory Commission 2003d, 3). The unresolved problems in this proceeding include, for example, so-called "seams issues" between regional markets with respect to market power mitigation,[84] among other matters.

As for monitoring and mitigation measures in the various regional markets, in Federal Energy Regulatory Commission (2004e), the Commission directed the CAISO to modify its recently-proposed "Enforcement Protocol" (see Table 2.1)—intended to complement the existing MMIP—to conform it to the Commission's earlier MBR Tariff Order (Federal Energy Regulatory Commission 2003b). In response, the CAISO has requested rehearing and clarification (California Independent System Operator 2004) of the Commission's Order. In the Midwest, the MISO's Market Monitoring and Mitigation Measures are one component of a recent tariff filing (Midwest ISO 2004), on which the Commission has yet to rule.

The model developed in the following chapters is motivated by the gaps in the theoretical foundations for market power assessment cited above, emphasizing the importance of electricity market architecture. Ultimately, this research should contribute insights to help clarify the ongoing market power debates reflected in the various administrative proceedings discussed here.

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[84] "[M]itigation tools which vary by region across market seams have the potential to create enforcement problems and undesirable behavioral incentives" (Federal Energy Regulatory Commission 2003d, 9).

> The sciences do not try to explain, they hardly even try to interpret, they mainly make models. By a model is meant a mathematical construct which, with the addition of certain verbal interpretations, describes observed phenomena. The justification of such a mathematical construct is solely and precisely that it is expected to work.
>
> —John von Neumann

> Electricity cannot be made fast, mortared up and ended, like London Monument, or the Tower, so that you shall know where to find it, and keep it fixed, as the English do with their things, forevermore; it is passing, glancing, gesticular; it is a traveller, a newness, a surprise, a secret, which perplexes them, and puts them out.
>
> —Emerson, *Essays and English Traits*

3 A supplier's forward market problem with financial contracts

THIS CHAPTER introduces the SF bidding model. We begin in section 3.1 below by introducing essential notation and terminology to develop the model of supplier behavior. Section 3.2 examines the nature of financial forward contracts and the cashflows that they introduce in market participants' optimization problems. Next, section 3.3 poses suppliers' forward market problem. Section 3.4 concludes by describing the backward induction solution algorithm for this problem.

3.1 The supply function bidding model: Notation and terminology

This section introduces nomenclature that we use to define the SF bidding model in the forward market; we develop this problem formally in section 3.3.

3.1.1 Timing and information structure of sequential markets

We interpret the multi-settlement SFE model as a two-stage game of complete[85] but imperfect information. In period 1, firms simultaneously formulate their forward market strategies—that is, their SF bids; this market clears at $t = 1$ with the revelation of the uncertain forward market demand function. Subsequently, in period 2, firms observe forward market outcomes and (again, simultaneously) formulate their SF bids for the spot market, which clears when the uncertain spot market demand function is revealed at $t = 2$. Finally, production takes place in period 3.[86] Figure 3.1 below highlights these features of the model:

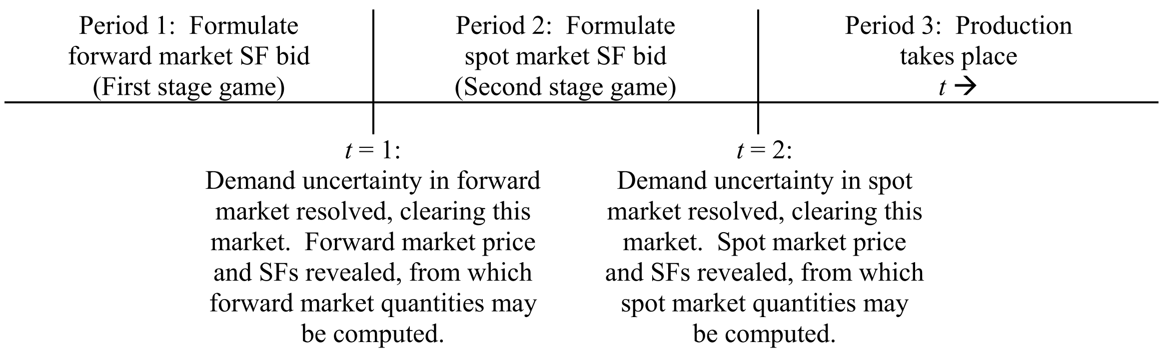

FIGURE 3.1: CONVENTIONS FOR THE TIMING OF FORWARD AND SPOT MARKET BIDDING IN A SINGLE ROUND OF THE MULTI-SETTLEMENT SFE MODEL

We consider only a single "round" of play, consisting of the following sequence of events (see Figure 3.1 above):

1. In period 1, supply-side market participants formulate an SF bid for the forward

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[85] Although—as explained below—demand in both the forward and spot markets is uncertain, firms' use of SFs as strategies and the existence of common prior probability distributions effectively offsets these two sources of uncertainty. See subsection 3.1.2 below on "Equilibrium concept" for further explanation.

[86] Since we assume that firms' commitments in the forward and spot markets are binding, firms do not face any additional decisions associated with production in period 3. Thus, we may neglect period 3 for the purposes of our analysis.

market. At the end of the period ($t = 1$), the market clears with the revelation of the forward demand function, which sets the forward market price. Also, firms' forward market SFs are revealed at this point, from which firms' forward market quantities may be computed.

2. The analogous sequence of events occurs in period 2 for the spot market.
3. Production occurs in period 3.

We assume that, as firms face forward market competition in Period 1, they begin the round with no contractual positions ex ante. Finally, if (in either market) a market-clearing price does not exist or is not unique, we assume that every firm then earns zero profits in that market.[87]

Although the single round of the game depicted in Figure 3.1 would in a typical competitive electricity market be repeated hourly, we abstract in this thesis—for simplicity—from what is, in reality, a repeated game. This is a strong simplification, as we thereby dispense with fundamental features of repeated games that are generally competitively significant. These include threat and punishment strategies and evolutionary phenomena such as learning and reputational effects. Nonetheless, the analysis of the static (two-stage) game is an essential first step toward more realistic models of behavior in what is, in reality, a dynamic setting.

The timing of the multi-settlement market game in Figure 3.1 reflects our assumption that firms can observe period 1 actions and outcomes before committing to

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[87] We make this assumption following KM (Klemperer and Meyer 1989, 1247 (n. 8)), who note that it ensures, in the single-market SFE model, that such outcomes do not arise in equilibrium. It is not a critical assumption, since the equilibria that they consider remain equilibria for reasonable alternative assumptions regarding firms' payoffs in the face of multiple equilibria. We expect that this will be the case, as well, for the multi-settlement SFE model examined here.

period 2 actions. This feature of observable actions and outcomes in a multi-stage game implies a *closed-loop information structure* (see Fudenberg and Tirole 1991, 130), in which players can condition their period 2 (spot market) play on period 1 (forward market) actions and outcomes; we call the corresponding strategies *closed-loop strategies*. In any *closed-loop equilibrium*,[88] firms' spot market bids given any forward market bids and outcomes must be a Nash equilibrium of the spot market stage game. When choosing their forward market bids, firms naturally recognize that optimal spot market bids will depend on forward market bids and outcomes (see Fudenberg and Tirole 1991, 132). Identifying the form of this dependence and its implications for the multi-settlement SFE model is a significant part of this chapter's analysis of the multi-stage game. Indeed, the closed-loop assumption is the natural information structure to associate with the multi-settlement SFE model. In this model (all the more so since in reality, this is a repeated game setting), firms will recognize that optimal spot market actions—for themselves and for their rivals—will depend on those in the forward market.

The (polar) alternative to the assumption of closed-loop strategies would be to assume *open-loop strategies*, which presuppose that players observe only their own history of play; accordingly, open-loop strategies depend only on time. Open-loop strategies are generally easier to analyze since they produce simpler optimality conditions (without intertemporal feedback terms) and since the open-loop strategy space is often much smaller. Open loop strategies are also often computed as benchmarks for examining strategic effects, that is, incentives to influence a rival's future actions through

[88] Adapting Fudenberg and Tirole's definition (1991, 131), we take a *closed-loop equilibrium* to mean a SPNE of a game in which players can (1) observe opponents' actions and realizations of uncertain parameters after each period, and (2) respond to these revelations in their future play.

one's own current actions (Fudenberg and Tirole 1991, 131). The open-loop assumption is less realistic in this information-rich environment, however, so that we use the closed-loop assumption exclusively in the present work.[89]

Assuming that firms' forward market SFs are perfectly observable may seem like a strong assumption; indeed, system operators do not simply announce these SFs in the course of market operation. There are several reasons, however, why observability may indeed be a plausible assumption within the context of competitive electricity markets: (1) the long history of economic regulation within the industry has generated a rich array of data and analyses concerning production technologies (specifically, costs) and demand forecasting; (2) the periodic nature of these markets provides an ideal environment for learning about competitors' short- and long-run strategies; and (3) market authorities customarily make market data publicly available (albeit with a few months' delay and usually in aggregate form) from which at least approximate models of the behavior of one's rivals might be inferred.

3.1.2 Equilibrium concept

Our use of *subgame perfection* as an equilibrium concept arises because of the sequential nature of the game depicted in Figure 3.1. In period 1, firms anticipate that the respective spot market SFs chosen later in period 2 will be in Nash equilibrium[90] with each other;

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[89] Beyond the extremes of open- and closed-loop strategies, a more flexible and arguably more realistic assumption regarding information structure would be *imperfect observability*. We save this case for future work, however, and focus in the present model on the benchmark case of perfectly observable actions. On the relationship between observability and strategic incentives in dynamic games, see the discussion of subsection 8.2.3.

[90] For now, assume that if there are multiple Nash equilibria in the spot market subgame, firms *successfully coordinate* on the particular spot market equilibrium to be anticipated (see n. 123). We address questions of equilibrium existence and uniqueness later in chapters 5 and 7.

maintaining this supposition, firms construct their forward market SFs. In a forward market equilibrium, these forward market SFs will themselves be in Nash equilibrium with each other (conditioned on the aforementioned spot market equilibrium). In period 2, firms choose their spot market SFs which will, in fact—as anticipated—constitute a Nash equilibrium in the spot market subgame. Finally, for simplicity, we consider only *pure strategy equilibria*.

As the solution of the forward market problem (see chapter 4) shows, firms' strategies depend, in general, on the probability distribution of spot market outcomes[91] and also on the relative profits associated with these outcomes. In contrast, firms' forward market actions are independent of the probability distribution of forward market outcomes. That is, as with the SFs in KM's single-market SFE model, forward market actions in the multi-settlement SFE model are ex post optimal in every state of the forward market. Since we will assume that all information is public (including, in particular, firms' costs and the aforementioned probability distributions), there is no incomplete information. Because we assume the use of closed-loop strategies with observable actions and outcomes, firms will respond optimally both to the realizations of random variables as well as to others' actions in previous periods. This condition is sufficient to permit the use of subgame perfection—in lieu of perfect Bayesian equilibrium (PBE)—as our equilibrium concept.[92] If, in contrast to this setting, firms'

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[91] See subsection 3.1.10 on "Demand functions" below for more on how uncertainty enters this problem.

[92] The equilibrium concept of PBE typically applies to multi-stage games of incomplete information. Because of incomplete information, the beliefs of players need to be characterized in equilibrium in addition to players' strategies. The PBE concept (Fudenberg and Tirole 1991, 326) consists, then, of a set of strategies and beliefs such that, at all times, (1) strategies are optimal given the beliefs and

actions were not perfectly observable or probability distributions of the uncertain parameters depended on subjective beliefs, then PBE might be the appropriate equilibrium concept.

3.1.3 *Industry structure and risk preferences*

We assume that the industry is a duopoly ($n = 2$), and that both producers are risk neutral. We index producers by $i = 1, 2$; unless otherwise specified, the index i ranges over these two values.

3.1.4 *Prices*

Let p^m be an arbitrary *price* in market m, where $m = f, s$ (for the forward and spot markets, respectively).[93] Denote an ex post *actual* (or *realized*) price in market m by a caret: $\hat{p}^m$.

3.1.5 *Supply functions*

The SFs that we consider for each market will map price (possibly together with other parameters, as will be discussed below) into quantity supplied by the firm in question. As in Klemperer and Meyer's (1989, 1250) analysis, we may intuitively characterize a firm's SF in a given market as the set of the firm's optimal[94] price-quantity points as its

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(2) the beliefs are obtained from equilibrium strategies and observed actions in accordance with Bayes' Rule.

[93] We use a superscript $m = f, s$ as an index on several market-specific variables and parameters to associate these with the forward and spot markets, respectively. The variable p^m is a scalar; later, using asterisks "*" to denote optimality, we will introduce optimal price *functions* p^{m*} (see chapter 4).

[94] As we will see below, we employ a distinct notion of optimality in each of the two markets in the model. Section 3.4—especially subsection 3.4.3—elaborates. For now, it suffices to interpret the quantities resulting from SFs simply as "optimal" in some meaningful sense.

residual demand function varies—due, say, to exogenous uncertainty in demand—assuming that its competitor's strategy is *fixed*. This property of SFs implies that, independent of the state of the world that is ultimately realized (we take p^m as a convenient state variable in the present discussion), the firm is guaranteed to supply the optimal quantity given this price, if, in the stage game in question, it chooses its SF as its action. By construction, therefore, SFs are *invariant* to the state of the world, and represent ex-post optimal actions *in every state of the world*.

The multi-settlement market framework, together with the SPNE concept, requires that we introduce some additional terminology to distinguish the various SF constructs that arise in this problem. The following discussion distinguishes SFs along various dimensions:

- *provisional* vs. *admissible* supply functions
- *imputed* vs. *optimal* supply functions
- *equilibrium* supply functions (applied only to optimal supply functions)

We next motivate and define each of these terms, and explain how these various types of SFs arise in the multi-settlement SFE model.

Provisional vs. admissible supply functions. This distinction arises in the multi-settlement SFE model due to our assumption of closed-loop strategies in SFs, but does not appear in the analogous single-market model of KM. For our two-market game, this distinction between provisional and admissible SFs applies only to spot market SFs. That is, for the spot market we have both "provisional" and "admissible" SFs, while for the forward market we have only "admissible" SFs.

Each firm conceives of its provisional spot market SF contemporaneously (in period 1) with the construction of its forward market SF. Conceptually, we may construct the *provisional* spot market SF for firm *i* via the following two-step process:

1. Fix a state of the world in the forward market and impute a spot market action to *j*.
2. Compute the (optimal) spot market SF for *i*.

Then, we repeat steps 1 and 2 above for every possible state of the world in the forward market. Each element of the set of *i*'s spot market SFs so computed is then a projection of firm *i*'s provisional spot market SF into the spot market price-quantity plane, indexed by the corresponding state of the world that generated it. We now denote firm *i*'s provisional spot market SF as $\Sigma_i^s\left(p^s;\bullet\right)$. In this notation, the subscript $i = 1, 2$ indexes firms and the superscripts *s* denote the spot market. The list of arguments for Σ_i^s, "$\left(p^s;\bullet\right)$," indicates that these arguments will include p^s in addition to other arguments characterizing the forward market outcome that remain to be determined. Thus, by this (incomplete) specification, the dimension of the domain of Σ_i^s will be greater than one. This fact is a reflection of the closed-loop property, discussed above, with which we have endowed these strategies. In order for period 2 actions to depend optimally on events in period 1, we must permit the arguments of Σ_i^s to reflect these period 1 events. Section 3.3 below will complete the specification of the arguments of $\Sigma_i^s\left(p^s;\bullet\right)$ appropriately for the closed-loop SPNE of the multi-settlement SFE model.

As Figure 3.1 depicts, in periods 1 and 2 firms formulate and submit to the market-clearing authority their forward and spot market SFs, respectively. In contrast to the provisional SFs discussed above, we define an *admissible* SF for firm i (or, equivalently, firm i's "bid") as any SF—in either market—that is consistent with exogenously specified *market rules* that determine the allowable form of bids. In game-theoretic terms, these market rules establish the *action space* that firms may use to participate in each market. Firms submit admissible SFs to the market-clearing authority; we denote such an admissible SF simply as $S_i^m\left(p^m\right)$.[95] Here, the subscript i again indexes firms, while the superscript $m = f, s$ now denotes the forward and spot markets, respectively. We assume that these market rules require firms' SF submitted bids to be twice continuously differentiable,[96] strictly increasing functions $S_i^m : \mathbb{R} \to \mathbb{R}$, so that $S_i^{m\prime}\left(p^m\right) > 0$. These functions map market m's clearing price p^m into the quantity $S_i^m\left(p^m\right)$ that the firm is willing to supply (or, in principle, purchase) at this price in market m.[97]

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[95] To preview the argument in subsection 3.4.3, given a parameterization and actual parameter values for optimal provisional spot market SFs $\Sigma_i^s\left(p^s;\bullet\right)$, optimal *admissible* and optimal *provisional* spot market SFs are related to each other, ex post, as $S_i^s\left(p^s\right) = \Sigma_i^s\left(p^s;\bullet\right) \,\forall\, p^s$.

[96] Piecewise differentiability (e.g., a piecewise linear spline, as in the (former) California PX; see note 46) or piecewise continuity (e.g., a step function) is a more likely bid restriction in actual electricity markets. We can, of course, approximate such functions arbitrarily closely almost everywhere with a continuously differentiable function, so we use the latter as an approximation of what "realistic" bids might look like.

[97] Whereby negative quantities would imply a net purchase, rather than a sale, by suppliers. As Klemperer and Meyer (1989, n. 12) explain, restricting firms in their model to choosing nonnegative quantities at all prices would yield the same results, but would complicate the analysis by permitting residual demand functions that are not everywhere differentiable. We similarly permit negative quantities in either market, in principle. In the specific numerical examples of chapter 7 (see, in particular, problems (7.58) and (7.61)), however, we exogenously restrict spot market equilibrium quantities $\overline{q}_i^s$ to be nonnegative, for simplicity. In contrast, forward market equilibrium quantities are not so restricted. Because the forward market is purely financial in nature (see section 3.2), negative forward market quantities are unproblematic and are not precluded in the multi-settlement SFE model. We will see in

The multi-settlement SFE model uses provisional spot market SFs *before* resolution of forward market uncertainty, and admissible spot market SFs *after* resolution of this uncertainty. To derive the provisional spot market SFs in Period 1, we use mathematical expectations to accommodate spot market uncertainty,[98] while we optimally account for forward market uncertainty via the forward market SFs. Later, in Period 2, forward market uncertainty has been resolved and we then derive admissible spot market SFs that are optimal given forward market actions and outcomes. Note that firms do not actually submit the optimal provisional spot market SFs to the market-clearing authority; we compute them solely because, as we argue in section 3.3 below, optimal admissible forward market SFs depend on them.

Imputed vs. optimal supply functions. This distinction arises in the multi-settlement market SFE model, and also in the single-market SFE model (e.g., that of KM). In determining the Nash equilibrium in SFs in each stage game of the multi-settlement SFE model, we posit that each firm assumes an SF—an *imputed* SF—on the part of its competitor, and then determines its own *optimal* SF given this assumption. This sequence of steps of imputation and optimization occurs once for each of our two markets: for the forward market in period 1, and for the spot market in period 2 (see Figure 3.1). So, for each firm and in each market, we will have both imputed and optimal

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chapter 7 that, given the slope restrictions $S_i^{f\prime}\left(p^f\right) > 0$ noted above for the forward market, forward market SFs over reasonable price ranges tend, in any event, to produce positive forward market quantities.

In principle, market institutions define the criteria for admissible SFs, imposing additional restrictions apart from increasingness—for example, piecewise linearity, minimum and maximum price levels, etc.—on their form. Beyond the above definition of S_i^m, we do not impose any such restrictions ex ante, but expect—as KM find—that certain properties characterizing equilibrium SFs will emerge endogenously.

[98] Eq. (3.35) in section 3.3 will make this notion more precise.

SFs. We denote imputed SFs with tildes "˜," and so write $\tilde{S}_i^m\left(p^m\right)$, $m = f, s$, for firm *i*'s imputed admissible SFs in market *m*. Similarly, $\tilde{\Sigma}_i^s\left(p^s;\bullet\right)$ denotes firm *i*'s imputed provisional spot market SF. For consistency with our assumed market rules, we assume here that the SFs that firms impute to their rivals will be strictly increasing in p^s, that is, $\tilde{S}_i^{m\prime}\left(p^m\right) > 0$ ($m = f, s$) and $\tilde{\Sigma}_i^{s\prime}\left(p^s;\bullet\right) > 0$.[99]

Equilibrium supply functions. We apply the modifier "equilibrium" to optimal SFs in either market that also constitute a Nash equilibrium—that is, a pair of optimal SFs, each of which is a best response to the other in all possible states of the world. We add an upper bar "¯" to the notation for an optimal SF to denote an *equilibrium* optimal SF. Thus in period 1, we may derive firm *i*'s *equilibrium optimal provisional spot market SF*, $\bar{\Sigma}_i^s\left(p^s;\bullet\right)$, and its *equilibrium optimal admissible forward market SF*, $\bar{S}_i^f\left(p^f\right)$ (assuming that such equilibria exist). Analogously, in period 2, we may derive firm *i*'s "equilibrium optimal admissible spot market SF," $\bar{S}_i^s\left(p^s\right)$ (again assuming existence). Finally, firm *i*'s *SPNE strategy* for the multi-settlement market SFE model consists of a set of SFs, one for each market, namely,

1. an equilibrium optimal admissible forward market SF, $\bar{S}_i^f\left(p^f\right)$
2. an equilibrium optimal provisional spot market SF, $\bar{\Sigma}_i^s\left(p^s;\bullet\right)$

[99] In the expression $\tilde{\Sigma}_i^{s\prime}\left(p^s;\bullet\right)$, the prime ("$\prime$") denotes differentiation with respect to the argument p^s.

That is, for the time being, we define (for now) the SPNE as follows:[100]

$$\left\{\bar{S}_i^f\left(p^f\right), \bar{\Sigma}_i^s\left(p^s;\bullet\right)\right\}, \quad i = 1, 2 \quad \Leftrightarrow \quad \begin{array}{c}\text{SPNE for the two-player,} \\ \text{multi-settlement market SFE game.}\end{array} \tag{3.1}$$

In solving the multi-settlement SFE model, the natural focus is on the constituent strategies of the SPNE (3.1). Thus, where we may economize on terminology without ambiguity, we suppress the descriptive modifiers "equilibrium," "optimal," and "admissible" applied to SFs. That is, we consider SFs to be "equilibrium and optimal SFs" unless otherwise specified. Accordingly, in the forward market, we generally refer to an

"equilibrium optimal admissible forward market SF"

as simply a

"forward market SF."

In the spot market, in contrast, we refer to an

"equilibrium optimal (provisional or admissible) spot market SF"

as simply a

"spot market SF."

Here, the provisional-admissible distinction should be clear from the context in which the spot market SF appears, and from the notation used. Nonetheless, for clarity in what

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100 We will revisit the SPNE's definition (3.1) in section 3.3 below, once the specification of $\bar{\Sigma}_i^s\left(p^s;\bullet\right)$ is complete.

follows, we add the modifiers "provisional" or "admissible" to describe spot market SFs where appropriate.

Table 3.1 below summarizes all of the distinctions among the various SF constructs introduced above.

TABLE 3.1: A TAXONOMY OF SUPPLY FUNCTIONS IN THE MULTI-SETTLEMENT SFE MODEL FOR FIRM i

Period 1	**Forward market stage game problem:** Each firm *i* formulates its forward market SF bid	
	Assumed exogenously: • Imputed provisional spot market SFs $\tilde{\Sigma}_i^s\left(p^s;\bullet\right)$ • Imputed admissible forward market SFs $\tilde{S}_i^f\left(p^f\right)$	Computed endogenously: • Optimal provisional spot market SFs $\Sigma_i^s\left(p^s;\bullet\right)$ • Optimal admissible forward market SFs $S_i^f\left(p^f\right)$
	Forward market stage game equilibrium actions: • Equilibrium optimal admissible forward market SFs $\overline{S}_i^f\left(p^f\right)$ ("forward market SFs"), assuming equilibrium optimal provisional spot market SFs $\overline{\Sigma}_i^s\left(p^s;\bullet\right)$ ("provisional spot market SFs")	
Period 2	**Spot market stage game problem:** Each firm *i* formulates its spot market SF bid	
	Assumed exogenously: • Imputed admissible spot market SFs $\tilde{S}_i^s\left(p^s\right)$	Computed endogenously: • Optimal admissible spot market SFs $S_i^s\left(p^s\right)$
	Spot market stage game equilibrium actions: • Equilibrium optimal admissible spot market SFs $\overline{S}_i^s\left(p^s\right)$ ("admissible spot market SFs")	
SPNE for the two-period game (multi-settlement market SFE): Sequence of equilibrium optimal SFs, one for each market: $\left\{\overline{S}_i^f\left(p^f\right),\overline{\Sigma}_i^s\left(p^s;\bullet\right)\right\}$		

3.1.6 Quantities

Define q_i^m as a *quantity* supplied by firm *i* in market *m*. This quantity is simply the firm's SF evaluated at some price in market *m*, that is, $q_i^m \equiv S_i^m\left(p^m\right)$. Using the imputed admissible SF for market m, we define a corresponding *imputed quantity* for firm i in market m, $\tilde{q}_i^m \equiv \tilde{S}_i^m\left(p^m\right)$. Similarly, define from market m's equilibrium optimal admissible SF the *equilibrium quantity*, $\overline{q}_i^m \equiv \overline{S}_i^m\left(p^m\right)$. Finally, we denote the ex post *actual* (or *realized*) quantity awarded to firm i in market m (not necessarily an equilibrium quantity) with a caret: $\hat{q}_i^m$.

3.1.7 Revenues

Let revenues of firm i in market m be R_i^m, so that

$$R_i^m = p^m q_i^m . \tag{3.2}$$

3.1.8 Cost functions

Let the *cost function* for firm i's production be $C_i\left(q_i^s\right)$ for $q_i^s \geq 0$ (whereby producers' cost functions may differ). We let this cost function pass through the origin, so that we consider only variable costs. Let $C_i\left(q_i^s\right)$ be twice differentiable (except perhaps at the origin) and be *common knowledge*. We assume that marginal cost $C_i'\left(q_i^s\right)$ is strictly increasing for positive quantities, that is, $C_i''\left(q_i^s\right) > 0$ for $q_i^s > 0$. We assume further that there are no capacity constraints on firm i's productive capacity; in other words, $C_i\left(q_i^s\right)$ remains finite for arbitrarily large q_i^s. Note that for simplicity, this formulation abstracts from the non-convexities introduced by start-up costs, no-load costs, and ramp rate limitations.

For any state of the world, the argument of firm i's cost function, q_i^s, is equal to firm *i*'s spot market residual demand function evaluated at the spot market clearing price, p^s.

3.1.9 *Profits*

We take profits in either market to mean *operating profits*, that is, short-term revenue less variable production costs. This convention treats all fixed costs as sunk and thus irrelevant to the present analysis.

3.1.10 *Demand functions*

This subsection considers first the spot market demand function, and then the forward market demand function.

We denote the *spot market demand function* as $D^s\left(p^s, \varepsilon^s\right)$, where we assume $\varepsilon^s \in E^s \subseteq \mathbb{R}$ to be an *additive* stochastic shock to demand in the spot market.[101] That is, we may write $D^s\left(p^s, \varepsilon^s\right)$ in additively separable form as

$$D^s\left(p^s, \varepsilon^s\right) = D_0^s\left(p^s\right) + \varepsilon^s, \tag{3.3}$$

where we refer to $D_0^s\left(p^s\right)$ as the *shape component* of spot market demand. Given $D^s\left(p^s, \varepsilon^s\right)$, define $D_0^s\left(p^s\right)$ as

$$D_0^s\left(p^s\right) \equiv D^s\left(p^s, \varepsilon^s\right) - D^s\left(0, \varepsilon^s\right) \tag{3.4}$$

such that $D_0^s\left(0\right) = 0$. That is, $D_0^s\left(p^s\right)$ passes through the origin of the p^s-q^s plane.[102] Combining eqs. (3.3) and (3.4), we also have that

[101] KM also rely upon this assumption, but relax it for some of their comparative statics analysis.

$$D^s\left(0, \varepsilon^s\right) \equiv \varepsilon^s , \tag{3.5}$$

implying that ε^s is the quantity-axis intercept of $D^s\left(p^s, \varepsilon^s\right)$. Let the support of ε^s, E^s, be an interval on the real line, $E^s \equiv \left[\underset{\smile}{\varepsilon}^s, \widehat{\varepsilon}^s\right]$, $\underset{\smile}{\varepsilon}^s < \widehat{\varepsilon}^s$. The upper limit of ε^s's support, $\widehat{\varepsilon}^s$, may be infinite, in which case $E^s = \left[\underset{\smile}{\varepsilon}^s, \infty\right)$. As with prices and quantities, let a caret "ˆ" denote the ex post *actual* (or *realized*) value of the shock ε^s, $\hat{\varepsilon}^s$. Figure 3.2 below illustrates the relationships in eqs. (3.3)–(3.5).

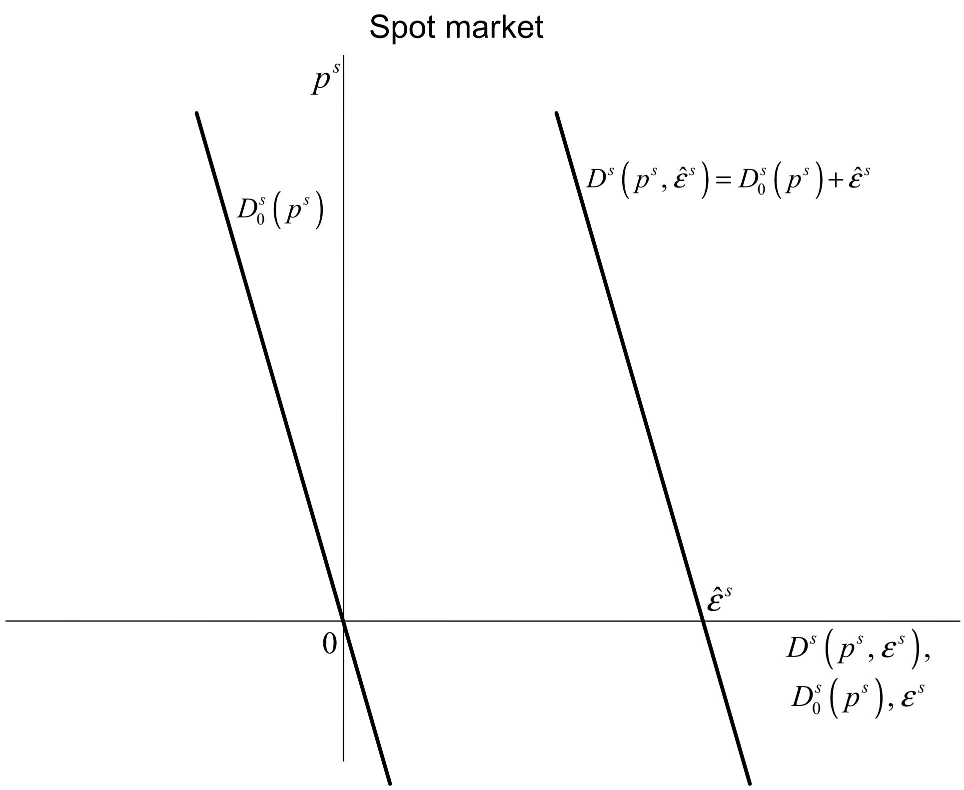

FIGURE 3.2: THE SPOT MARKET DEMAND FUNCTION $D^s\left(p^s, \varepsilon^s\right)$ FOR $\varepsilon^s = \hat{\varepsilon}^s$, AND THE SHAPE COMPONENT OF SPOT MARKET DEMAND, $D_0^s\left(p^s\right)$

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[102] The shock ε^s also shares this origin.

As an example, suppose that $D^s\left(p^s, \varepsilon^s\right) = -0.01p^s + \varepsilon^s$. Then, we would have that $D_0^s\left(p^s\right) \equiv -0.01p^s$, and as required, $D_0^s\left(0\right) = 0$.

The assumed functional form (3.3) for $D^s\left(p^s, \varepsilon^s\right)$ has important implications for the analysis. First, following KM, the additive shock ε^s shifts—but does not rotate—the spot market demand function $D^s\left(p^s, \varepsilon^s\right)$, and so we have that its cross-partial derivative is zero, that is,

$$\frac{\partial^2 D^s\left(p^s, \varepsilon^s\right)}{\partial p^s \partial \varepsilon^s} = 0. \tag{3.6}$$

Second, it also follows from eq. (3.3) that[103]

$$D^{s\prime}\left(p^s, \varepsilon^s\right) \equiv \frac{\partial D^s\left(p^s, \varepsilon^s\right)}{\partial p^s} = D_0^{s\prime}\left(p^s\right) \quad \forall\, p^s, \varepsilon^s, \tag{3.7}$$

so that the derivatives $D^{s\prime}\left(p^s, \varepsilon^s\right)$ and $D_0^{s\prime}\left(p^s\right)$ are interchangeable. In chapter 6, we will show endogenously that spot market demand is *downward-sloping*, that is, $D^{s\prime}\left(p^s, \varepsilon^s\right) \equiv \partial D^s\left(p^s, \varepsilon^s\right) \big/ \partial p^s < 0$.

The spot market demand function, $D^s\left(p^s, \varepsilon^s\right)$, arises because of final consumers' willingness to pay for energy-related services (e.g., for either consumptive or productive purposes) that electricity can provide. Subsection 6.6.1 explains how consumers' utility

[103] With a slight abuse of notation, we use a prime (" $'$ ") on the spot market demand function to indicate partial differentiation with respect to price. As we do not need to refer to the partial derivative with respect to the stochastic shock ε^s, there is no ambiguity.

functions give rise endogenously to $D^s\left(p^s,\varepsilon^s\right)$. We also assume $D_0^s\left(p^s\right)$ to be common knowledge and that the shock ε^s (due, for example, to varying weather conditions, economic activity or other effects on consumption) is drawn from an exogenous, common knowledge distribution. From chapter 5 onward, we restrict the analysis and consider a *simplified affine example*, in which we assume that the spot market demand function is *affine*.

Consider now the *forward market demand function*, which we denote as $D^f\left(p^f,\varepsilon_0^f\right)$. Similar to the spot market analysis, we assume $\varepsilon_0^f \in E^f \subseteq \mathbb{R}$ to be an *additive* stochastic shock to demand in the forward market. That is, we may write $D^f\left(p^f,\varepsilon_0^f\right)$ in additively separable form as

$$D^f\left(p^f,\varepsilon_0^f\right) = D_0^f\left(p^f\right) + \varepsilon_0^f, \tag{3.8}$$

where we refer to $D_0^f\left(p^f\right)$ as the *shape component* of forward market demand (which we define in eq. (3.9) below). As we show in chapter 6, in contrast to the situation in the spot market, the forward market demand function $D^f\left(p^f,\varepsilon_0^f\right)$ is endogenous to the forward market SFs $\overline{S}_i^f\left(p^f\right)$. The properties of $D^f\left(p^f,\varepsilon_0^f\right)$ therefore depend on the properties of $\overline{S}_i^f\left(p^f\right)$; moreover, the definition of $D^f\left(p^f,\varepsilon_0^f\right)$ is somewhat more involved than the definition of $D^s\left(p^s,\varepsilon^s\right)$ above.

Before discussing further the properties of the functions in eq. (3.8), consider the forward market SFs, $\overline{S}_i^f\left(p^f\right)$. For a variety of reasons, it may be the case that, beginning from a given initial condition, we cannot define a forward market SF $\overline{S}_i^f\left(p^f\right)$

over all prices $p^f \in \mathbb{R}$. Rather, the SF may have a restricted domain of definition, say, from some minimum price $\underset{\smile}{p}^f$ to a maximum price $\widehat{p}^f > \underset{\smile}{p}^f$. In this case, the *domain* of firm *i*'s equilibrium forward market SF $\bar{S}_i^f\left(p^f\right)$ is the interval $\left[\underset{\smile}{p}^f, \widehat{p}^f\right]$; we refer to this interval as a *domain restriction* on the function $\bar{S}_i^f\left(p^f\right)$.[104] Because it is endogenous, the forward market demand function $D^f\left(p^f, \varepsilon_0^f\right)$ inherits $\bar{S}_i^f\left(p^f\right)$'s domain restrictions. Assume, therefore, that both SFs $\bar{S}_i^f\left(p^f\right)$—and hence also $D^f\left(p^f, \varepsilon_0^f\right)$—are defined over the interval $\left[\underset{\smile}{p}^f, \widehat{p}^f\right]$.

Now assume some *reference price* $p_0^f \in \left[\underset{\smile}{p}^f, \widehat{p}^f\right]$ contained in the interval over which demand is defined. In the following, we define the demand shock ε_0^f in eq. (3.8) so that it is equal to the demand function evaluated at the reference price p_0^f. To do this, assume a function $D^f\left(p^f, \varepsilon_0^f\right)$ as in eq. (3.8), and define the shape component of the forward market demand function $D_0^f\left(p^f\right)$ as

$$D_0^f\left(p^f\right) \equiv D^f\left(p^f, \varepsilon_0^f\right) - D^f\left(p_0^f, \varepsilon_0^f\right) \tag{3.9}$$

[104] Such domain restrictions may arise for a variety of theoretical or practical reasons as the analysis of chapter 7 makes clear. As an example of the former, it may be the case that, as we move along a specific SF for a particular firm, that firm's second-order condition (SOC) for profit maximization may be violated for prices above or below a certain level. The SF may not be continued into the region in which the SOC does not hold; to prevent this, the domain of the SF must be restricted accordingly. Alternatively, it may be that the firms' SF becomes downward-sloping in p^f over certain price ranges. An example of a practical reason for a domain restriction arises in chapter 7. There, we see that the presence of singularities may limit the range of prices over which we are able to successfully numerically integrate the conditions characterizing the forward market SFs.

such that $D_0^f\left(p_0^f\right)=0$. That is, $D_0^f\left(p^f\right)$ passes through the point $\left(p^f, q^f\right)=\left(p_0^f, 0\right)$.[105] Combining eqs. (3.8) and (3.9), we also have that

$$D^f\left(p_0^f, \varepsilon_0^f\right) \equiv \varepsilon_0^f \, . \tag{3.10}$$

Let the support of ε_0^f, E^f, be an interval on the real line, $E^f \equiv \left[\underset{\smile}{\varepsilon}_0^f, \overset{\frown}{\varepsilon}_0^f\right]$, $\underset{\smile}{\varepsilon}_0^f < \overset{\frown}{\varepsilon}_0^f$. The upper limit of ε_0^f's support, $\overset{\frown}{\varepsilon}_0^f$, may be infinite, in which case $E^f = \left[\underset{\smile}{\varepsilon}_0^f, \infty\right)$. Again, let a caret "ˆ" denote the ex post *actual* (or *realized*) value of the shock ε_0^f, $\hat{\varepsilon}_0^f$.[106] Figure 3.3 below illustrates the relationships in eqs. (3.8)–(3.10) where, for concreteness and ease of exposition, the figure assumes that

$$p_0^f = \underset{\smile}{p}^f \, , \tag{3.11}$$

though as noted above, any $p_0^f \in \left[\underset{\smile}{p}^f, \overset{\frown}{p}^f\right]$ is a suitable choice.

[105] The shock ε_0^f also has its origin at $q^f = 0$.

[106] The use in the forward market of an arbitrary reference price p_0^f is a generalization of the approach used for the spot market analysis above. There, the spot market reference price is simply *zero*, for simplicity (compare, for example, eqs. (3.5) and (3.10)). The affine functional form of $D^s\left(p^s, \varepsilon^s\right)$ assures us that for finitely-sloped functions $D^s\left(p^s, \varepsilon^s\right)$, this function will intersect the quantity axis (at ε^s).

The subscript "$_0$" on ε_0^f indicates that the forward market demand shock is defined *relative to* the reference price p_0^f.

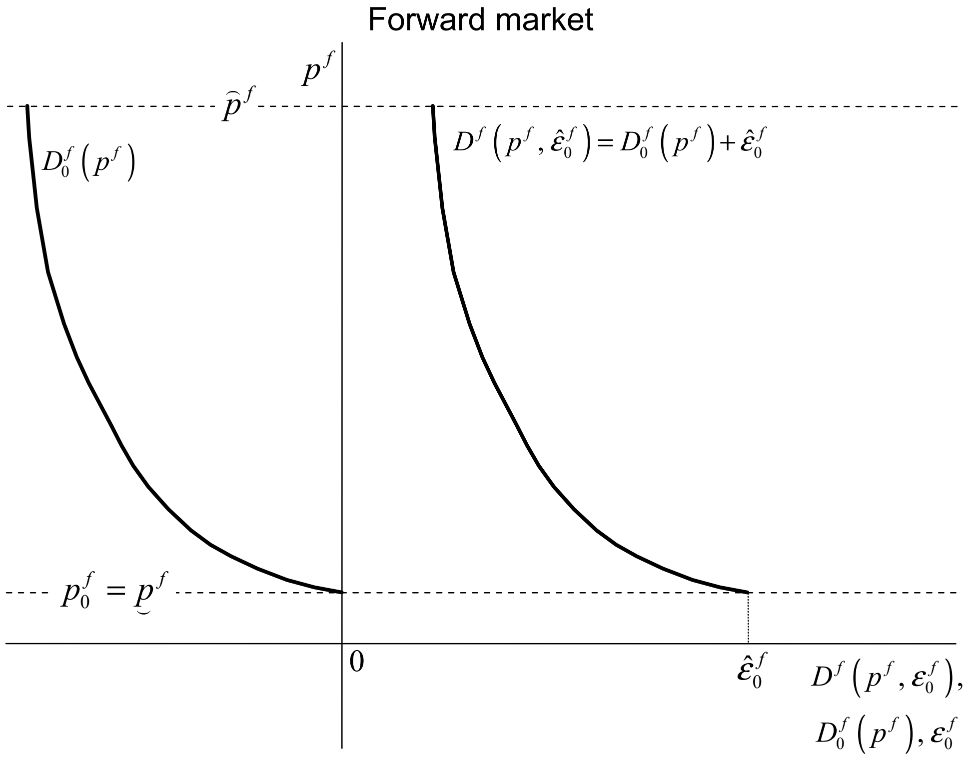

FIGURE 3.3: THE FORWARD MARKET DEMAND FUNCTION $D^f\left(p^f,\varepsilon_0^f\right)$ DEFINED ON $\left[\underset{\smile}{p}^f,\widehat{p}^f\right]$ FOR $\varepsilon_0^f=\hat{\varepsilon}_0^f$ AND THE SHAPE COMPONENT OF SPOT MARKET DEMAND $D_0^f\left(p^f\right)$, TAKING REFERENCE PRICE p_0^f TO BE EQUAL TO $\underset{\smile}{p}^f$

We may give an example analogous to that used in the discussion of spot market demand. Namely, suppose that $D^f\left(p^f,\varepsilon_0^f\right)=e^{-\left(p^f-p_0^f\right)}-1+\varepsilon_0^f$. Then, we would have that $D_0^f\left(p^f\right)\equiv e^{-\left(p^f-p_0^f\right)}-1$, and as required, $D_0^f\left(p_0^f\right)=e^{-\left(p_0^f-p_0^f\right)}-1=0$.

The assumed functional form (3.8) for $D^f\left(p^f,\varepsilon_0^f\right)$ has important implications for the analysis. First, following KM, the additive shock ε_0^f shifts—but does not rotate—the forward market demand function $D^f\left(p^f,\varepsilon_0^f\right)$, and so we have that its cross-partial derivative is zero, that is,

$$\frac{\partial^2 D^f\left(p^f, \varepsilon_0^f\right)}{\partial p^f \partial \varepsilon_0^f} = 0\,. \tag{3.12}$$

Second, it also follows from eq. (3.8) that[107]

$$D^{f\prime}\left(p^f, \varepsilon_0^f\right) \equiv \frac{\partial D^f\left(p^f, \varepsilon_0^f\right)}{\partial p^f} = D_0^{f\prime}\left(p^f\right) \quad \forall\, p^f, \varepsilon_0^f\,, \tag{3.13}$$

so that the derivatives $D^{f\prime}\left(p^f, \varepsilon_0^f\right)$ and $D_0^{f\prime}\left(p^f\right)$ are interchangeable. In chapter 6, we will show endogenously that forward market demand is *downward-sloping* under our assumptions, that is, $D^{f\prime}\left(p^f, \varepsilon_0^f\right) \equiv \partial D^f\left(p^f, \varepsilon_0^f\right) \big/ \partial p^f < 0$.

As noted above, the forward market demand function, $D^f\left(p^f, \varepsilon_0^f\right)$, is endogenous in the multi-settlement SFE model. Forward market demand arises due to the market activity of risk-averse consumers in an uncertain environment, who seek to buy forward contracts for electricity given spot market demand $D^s\left(p^s, \varepsilon^s\right)$. We assume that $D_0^f\left(p^f\right)$ is common knowledge. Later, chapter 6 provides a systematic analysis of the provenance of the forward market demand function in the multi-settlement SFE model (including the distribution of ε_0^f), and confirms that $D^f\left(p^f, \varepsilon_0^f\right)$ indeed has the properties discussed here.

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[107] Similar to the notation in the spot market, we use a prime (" $\prime$ ") on the forward market demand function to indicate partial differentiation with respect to price. As we do not need to refer to the partial derivative with respect to the stochastic shock ε_0^f, there is no ambiguity.

3.2 The nature of financial forward contracts

The forward contracts considered in the multi-settlement SFE model are purely *financial* in the sense that forward market positions neither commit firms to a particular physical production schedule, nor commit purchasers to consume electricity. Rather, these financial contracts represent property rights to a cash flow based on (1) contract quantity and (2) relative prices in the forward and spot markets.[108] Firms may liquidate their forward contract positions partially or completely in the spot market by repurchasing the desired level of output at the spot market price.[109] Consistent with this definition, in the analytical model developed in this section, forward contract positions $\hat{q}_i^f$ do not directly enters firms' cost functions. Rather, as the multi-settlement SFE model will make clear, firm *i*'s spot market quantity produced, $\hat{q}_i^s$, depends, through $\bar{\Sigma}_i^s$, on the forward market quantities $\hat{q}_1^f$ and $\hat{q}_2^f$.

In a given round of the multi-settlement market, we define the net cash flow CF_i *to* firm *i* from a financial forward contract sold by firm *i* as

$$CF_i = \left(p^f - p^s\right)q_i^f . \tag{3.14}$$

In eq. (3.14), each factor $\left(p^f - p^s\right)$ and q_i^f in CF_i—and hence CF_i itself—may be positive, negative, or zero. Thus, if $CF_i > 0$ in a given round of the multi-settlement

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[108] We assume these property rights to be perfectly and costlessly enforceable.

[109] Many electricity forward markets reflect this property: at least at trading "hubs," these markets tend to be liquid, offering reliable resale opportunities.

market, then contract holders pay CF_i to firm *i*. If, in contrast, $CF_i < 0$, then firm *i* pays CF_i to contract holders.

The literature on electricity markets commonly refers to this form of contract as a two-way *contract for differences*, or *CFD*, where the term "differences" denotes, naturally, the difference between the contract (or forward market) price, p^f, and the spot price, p^s. A CFD is a simple financial instrument designed to enable market participants to lock in a certain price in the forward market for a quantity of electricity. If exactly the forward contract quantity is transacted in the spot market, then the financial outcome of that market round is *independent* of a (usually more volatile) spot market price.[110] The bid-based forward market examined in this investigation is essentially a double auction for CFDs, with (in principle) both demand[111] and supply bidding to transact different quantities, depending on price.

To focus attention on this essential feature of the CFD, it is helpful to consider separately the three possible outcomes from firm *i*'s perspective: (1) firm *i* is *under-contracted* ($q_i^f < q_i^s$), (2) firm *i* is *fully contracted* ($q_i^f = q_i^s$), and (3) firm *i* is *over-contracted* ($q_i^f > q_i^s$). We examine, in turn, each of these outcomes below, offering an intuitive interpretation of each transaction:

[110] On this point, see paragraph 2 below and also Borenstein et al. (2000, 4ff.).

[111] Although development of an active bid-based demand side within competitive electricity markets has historically lagged behind that of the supply side, provisions for price-sensitive bids by demand-side agents are in place in many markets around the world (see, e.g., International Energy Agency 2001, 83).

1. If $0 < q_i^f < q_i^s$, we may interpret the CFD as a fixed-price contract under which firm *i* and consumers transact the first q_i^f of *i*'s output at p^f. Market participants then transact the remaining portion of *i*'s spot market output, $q_i^s - q_i^f$, at p^s.[112]

2. If $q_i^f = q_i^s$, we may interpret the CFD as a fixed-price contract under which firm *i* and consumers transact *i*'s entire output of q_i^s at p^f.

3. If $q_i^f > q_i^s$, we may interpret the CFD as a fixed-price contract under which firm *i* and consumers transact *i*'s output of q_i^s at p^f. Consumers then buy out of their remaining contractual commitment of $q_i^f - q_i^s$ at a price of $p^f - p^s$, that is, the demand side makes a buy-out payment of $\left(p^f - p^s\right)\left(q_i^f - q_i^s\right)$ to firm *i*.[113]

   Alternatively, we may view this buy-out payment as two separate transactions. Under this interpretation, the demand side first takes title to its remaining contractual commitment of $q_i^f - q_i^s$ through a payment of $p^f\left(q_i^f - q_i^s\right)$ (thereby fulfilling the forward contract). The demand side then resells this unwanted quantity on the spot market at the market-clearing price, thereby receiving a payment $p^s\left(q_i^f - q_i^s\right)$.

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[112] If, instead, we have that $q_i^f < 0 < q_i^s$, the interpretation of the associated transaction (though not the basic arithmetic) changes somewhat. Namely, in this case, we may interpret the CFD as a fixed-price contract under which firm *i* purchases q_i^f forward contracts from consumers at p^f. Market participants then transact the quantity $q_i^s - q_i^f$ ($> q_i^s$) at p^s.

[113] In this scenario, if $\left(p^f - p^s\right) < 0$, firm *i* will *pay* demand side participants to reduce their consumption below the contracted quantity. That is, the "buy-out" payment to firm *i* given by the product $\left(p^f - p^s\right)\left(q_i^f - q_i^s\right)$ will be negative.

In practice, because the forward and spot markets clear at distinct points in time, a supplying firm *i* perceives the cash flow CF_i (see eq. (3.14)) from the forward contract as comprising two separate components. Namely, firm *i* first experiences an inflow (assuming $p^f > 0$) of $p^f q_i^f$ once the forward market clears at $t = 1$. Equation (3.2) denoted this term as firm *i*'s forward market revenue, R_i^f, given by

$$R_i^f = p^f q_i^f , \tag{3.15}$$

which is firm *i*'s cash flow in the forward market. Next, once the spot market clears at $t = 2$, firm *i* incurs a contract *settlement payment* of

$$p^s q_i^f . \tag{3.16}$$

This settlement payment is one component of firm *i*'s cash flows in the spot market (see the following section for more details). Together, the difference of R_i^f in eq. (3.15) and $p^s q_i^f$ in (3.16) is equal to CF_i from eq. (3.14).

We refer hereinafter to "(financial) forward contracts," "forward contracting," etc. with the understanding that such contracts have the structure of CFDs as detailed in this section.

3.3 Posing the forward market problem

To pose firm *i*'s forward market problem in the multi-settlement market SFE model with forward contracting, it will be useful to begin by considering firm *i*'s action in the spot market, and work backward from there. This approach reflects the solution algorithm of backward induction which we employ later in section 3.4.

Recall from subsection 3.1.1 that the closed-loop information structure posited for our problem implies that firms are able to condition their spot market play on forward market actions and outcomes. Accordingly, firms recognize when choosing their forward market bids that, ultimately, spot market bids will depend on those in the forward market. This observation motivated the definition of firm *i*'s provisional spot market SF, $\Sigma_i^s\left(p^s;\bullet\right)$, as its period 1 characterization of its later spot market action.[114]

In the multi-settlement SFE model, firm *i* is aware that the closed-loop information structure applies, as well, to its competitor, firm *j*. Therefore, the particular spot market SF that firm *i* *imputes* in period 1 to firm *j* when solving its (firm *i*'s) own forward market problem will likewise be a provisional spot market SF. In subsection 3.1.5, we denoted this SF as $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$ and assume it to be strictly increasing in p^s. Given this imputation, firm *i* will conceive of its spot market residual demand function as spot market demand, $D^s\left(p^s,\varepsilon^s\right)$, less firm *j*'s imputed provisional SF, $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$. In any spot market-clearing equilibrium, then, firm *i*'s spot market quantity q_i^s will lie on this residual demand function at the market-clearing price p^s. Therefore, we may define, for any arbitrary ε^s and corresponding market-clearing p^s,[115]

$$q_i^s \equiv D^s\left(p^s,\varepsilon^s\right)-\tilde{\Sigma}_j^s\left(p^s;\bullet\right). \tag{3.17}$$

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[114] In that discussion, the placeholder "$\bullet$" in the argument list of Σ_i^s represented the (as-yet-unspecified) influence of the forward market outcome on firm *i*'s spot market action. Later in this section, we will be able to identify these unknown arguments from the specification of the problem's objective function.

[115] Note that we have defined q_i^s, for our present purposes, as a *point*, not as a *function*. In section 4.1 below, we use a refinement of eq. (3.17) to construct a provisional spot market SF for firm *i*.

Now consider firm i's profits in the spot market, in the presence of financial forward contracts.[116] Section 3.2's discussion concerning these contracts highlighted one component of these profits, namely, the contract settlement payment, $p^s q_i^f$ (see expression (3.16)), paid by supplier firms (for $p^s q_i^f > 0$) to consumers. There are two more contributions to firm i's spot market profits, namely, *revenues* from sales of spot market output, and the *production cost* of spot market output itself. In subsection 3.1.7, we defined firm i's spot market revenues, R_i^s, as (see eq. (3.2))

$$R_i^s = p^s q_i^s, \tag{3.18}$$

and in subsection 3.1.8, denoted firm i's production cost as

$$C_i\left(q_i^s\right) \tag{3.19}$$

for its spot market quantity, q_i^s.

Before bringing together the three constituent terms of firm i's profits in the spot market—R_i^s, $p^s q_i^f$, and $C_i\left(q_i^s\right)$ from expressions (3.18), (3.16), and (3.19), respectively—consider again the above definition of firm i's equilibrium spot market quantity, q_i^s, that enters eqs. (3.18) and (3.19). Recall that eq. (3.17) defined the quantity q_i^s in terms of firm j's imputed provisional spot market SF, $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$. The spot market profits computed using this expression for q_i^s is—like the optimal admissible spot market SF—necessarily contingent on the realized forward market outcome. Until we observe

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[116] Green (1999a) studies the interaction of contract and spot markets (as subsection 1.5.2 discusses).

this realized forward market outcome, we may only express firm *i*'s spot market profits on a "provisional" basis, as well. For this reason, we refer to this notion of spot market profits for firm *i* as *firm* i*'s provisional spot market profits* given an imputed provisional spot market SF for firm *j*, $\tilde{\Sigma}_j^s$, and denote this as $\tilde{\pi}_i^s$, which we may write from expressions (3.18), (3.16), and (3.19) as[117]

$$\tilde{\pi}_i^s = R_i^s - p^s q_i^f - C_i\left(q_i^s\right). \tag{3.20}$$

In other words, $\tilde{\pi}_i^s$ in eq. (3.20) is firm *i*'s period 1 conception—that is, as it formulates its *forward* market bid—of its *spot* market profits.

Substituting for R_i^s from eq. (3.18), eq. (3.20) becomes

$$\tilde{\pi}_i^s = p^s q_i^s - p^s q_i^f - C_i\left(q_i^s\right). \tag{3.21}$$

Using eq. (3.17) to substitute for q_i^s in eq. (3.21) and including the functional arguments of $\tilde{\pi}_i^s$ yields

$$\begin{aligned} &\tilde{\pi}_i^s\left\{p^s, \tilde{\Sigma}_j^s\left(p^s;\bullet\right), q_i^f, \varepsilon^s\right\} \\ &\qquad = p^s \cdot \left[D^s\left(p^s,\varepsilon^s\right) - \tilde{\Sigma}_j^s\left(p^s;\bullet\right)\right] - p^s q_i^f - C_i\left[D^s\left(p^s,\varepsilon^s\right) - \tilde{\Sigma}_j^s\left(p^s;\bullet\right)\right]. \end{aligned} \tag{3.22}$$

The SF $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$ is arbitrary at this point and is therefore included as an argument of $\tilde{\pi}_i^s$ in eq. (3.22). The cost function and the spot market demand function are exogenously

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[117] The tilde "˜" on $\tilde{\pi}_i^s$ signifies that this profits expression is a function of the imputed SF $\tilde{\Sigma}_j^s$, which also bears a tilde.

fixed[118] throughout the analysis, and hence are not explicitly represented as arguments of $\tilde{\pi}_i^s$.

We may now characterize firm *i*'s spot market optimum given firm *j*'s imputed provisional spot market SF. Equation (3.22) gives an expression for firm *i*'s provisional spot market profits. For a Nash equilibrium in the spot market subgame in any state of the world, a necessary condition is that for any given demand shock ε^s, forward market quantity q_i^f, and imputed provisional spot market SF $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$ for firm *j*, firm *i* will choose an optimal—that is, "provisional spot market profit-maximizing"—price, $p^s = p_i^{s*}$, in the spot market.[119] Let the *optimal provisional spot market profits* for firm *i*, $\tilde{\pi}_i^{s*}$, be the maximized value of $\tilde{\pi}_i^s$ at p_i^{s*}, that is,

$$\tilde{\pi}_i^{s*}\left\{\tilde{\Sigma}_j^s\left(\bullet;\bullet\right), q_i^f, \varepsilon^s\right\} = \max_{p^s} \tilde{\pi}_i^s\left\{p^s, \tilde{\Sigma}_j^s\left(p^s;\bullet\right), q_i^f, \varepsilon^s\right\},$$

or, substituting from eq. (3.22) for $\tilde{\pi}_i^s$ in the above equation,

$$\begin{aligned}&\tilde{\pi}_i^{s*}\left\{\tilde{\Sigma}_j^s\left(\bullet;\bullet\right), q_i^f, \varepsilon^s\right\} = \\ &\qquad \max_{p^s}\left\{p^s\cdot\left[D^s\left(p^s,\varepsilon^s\right)-\tilde{\Sigma}_j^s\left(p^s;\bullet\right)\right]-p^s q_i^f - C_i\left[D^s\left(p^s,\varepsilon^s\right)-\tilde{\Sigma}_j^s\left(p^s;\bullet\right)\right]\right\}.\end{aligned} \tag{3.23}$$

Let us now specify the arguments of the SFs $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$, $j=1,2$. Recall that we wrote eq. (3.23) for a generic firm *i*'s spot market optimum, given a provisional spot

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[118] That is, the demand function $D^s\left(p^s,\varepsilon^s\right)$ is fixed up to the stochastic shock ε^s, which we do include as an argument of $\tilde{\pi}_i^s$.

[119] For now, we simply *assume* the existence of a unique equilibrium price p_i^{s*} for each ε^s; we consider this issue more formally in section 4.1.

market SF for firm j, $\tilde{\Sigma}_j^s$ ($i, j = 1, 2;\ i \neq j$). While we provide a more precise argument in section 3.4 and chapter 4 below, we argue—intuitively, at this point—as follows. A necessary condition for the SFs $\tilde{\Sigma}_1^s\left(p^s;\bullet\right)$ and $\tilde{\Sigma}_2^s\left(p^s;\bullet\right)$ to constitute a Nash equilibrium in the spot market subgame will be to satisfy eq. (3.23) for firms $i, j = 1, 2$ ($i \neq j$), given any forward market outcomes q_i^f *and* for any realization of the spot market demand shock ε^s. We are now in a position to ask, on what additional variables or parameters, apart from p^s, does j's imputed provisional spot market SF, $\tilde{\Sigma}_j^s$, depend? By inspection of the right-hand side of eq. (3.23), there are two possibilities: the demand shock, ε^s, and the forward quantity, q_i^f; we consider both of these parameters below.

Looking first at ε^s, we may rule this parameter out as a candidate for inclusion as an argument of $\tilde{\Sigma}_j^s$ with the following reasoning. From the taxonomy of SFs in subsection 3.1.5, the projection of $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$ into the p^s-q^s plane is $\tilde{S}_j^s\left(p^s\right)$, which has only p^s, and not ε^s, as an argument (that is, $\tilde{S}_j^s\left(p^s\right)$ is simply a continuous function in the p^s-q^s plane). The property that equilibrium SFs yield ex post optimal quantities in all states of the world[120] implies that $\tilde{S}_j^s$—and hence $\tilde{\Sigma}_j^s$—must be optimal for all ε^s and for all forward market outcomes. Thus, while p_i^{s*} will be (as argued above) a function of ε^s, $\tilde{S}_j^s$—and hence $\tilde{\Sigma}_j^s$—will not be functions of ε^s. We conclude that we must *not* include ε^s as an argument of $\tilde{\Sigma}_j^s$.

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[120] See Klemperer and Meyer (1989, 1250), and section 4.1 of the present investigation.

The forward quantity, q_i^f, also appears as a parameter on the right-hand side of eq. (3.23). The quantities q_i^f incorporate information about both (1) firms' forward market actions (i.e., their SF bids) and (2) the realization of forward market uncertainty, ε_0^f, while containing no definitive information about the spot market outcome, ε^s. Under our assumption of a closed-loop information structure in the multi-settlement SFE model (see subsection 3.1.1), firms can—and indeed, to ensure ex post optimality in the spot market, *must*—condition their spot market play on forward market actions and outcomes. They do so by incorporating the appropriate parameters from the forward market as arguments of their spot market SFs. From eq. (3.23), the appropriate forward market parameters are precisely the forward market quantities q_i^f. Because we impose eq. (3.23) for $i, j = 1, 2$ $(i \neq j)$ in equilibrium, we must include both firms' forward market quantities[121] in each function $\tilde{\Sigma}_1^s\left(p^s;\bullet\right)$ and $\tilde{\Sigma}_2^s\left(p^s;\bullet\right)$. In general, therefore, we write $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$ with its complete list of arguments as

$$\tilde{\Sigma}_j^s\left(p^s;\tilde{q}_j^f, q_i^f\right), \quad i, j = 1, 2;\ i \neq j, \tag{3.24}$$

[121] For consistency, note in firm *i*'s problem that, just as firm *i* imputes $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$ to firm *j* for the *spot market*, firm *i* will also impute $\tilde{S}_j^f\left(p^f\right)$ (as we argue below) to firm *j* in the *forward market*. In analyzing the provisional spot market equilibrium, however, we do not require firm *j*'s imputed forward market SF, but simply firm *j*'s forward market *quantity*, as imputed by firm *i*. We denote this quantity, which we have called firm *j*'s *imputed forward market quantity* (imputed by firm *i*) as $\tilde{q}_j^f$, given by $\tilde{q}_j^f = \tilde{S}_j^f\left(p^f\right)$. It is this quantity, $\tilde{q}_j^f$, that we include as one of the additional arguments of $\tilde{\Sigma}_j^s\left(p^s;\bullet\right)$, along with firm *i*'s own optimal forward market quantity q_i^f.

from now on, that is, $\tilde{\Sigma}_j^s : \mathbb{R}^3 \to \mathbb{R}$. Having specified the arguments of $\tilde{\Sigma}_j^s$, we restate eq. (3.23) using the parameterization of expression (3.24),

$$\tilde{\pi}_i^{s*}\left\{\tilde{\Sigma}_j^s\left(\bullet;\tilde{q}_j^f,q_i^f\right),q_i^f,\varepsilon^s\right\} = \max_{p^s}\left\{p^s\cdot\left[D^s\left(p^s,\varepsilon^s\right)-\tilde{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)\right]-p^s q_i^f \right. \\ \left. -C_i\left[D^s\left(p^s,\varepsilon^s\right)-\tilde{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)\right]\right\}, \tag{3.25}$$

and continue with the construction of firm i's forward market problem.

Given that both firms 1 and 2 maximize their provisional spot market profits (i.e., solve eq. (3.25)), we may state jointly necessary and sufficient conditions for a (pure strategy) Nash equilibrium in provisional spot market SFs:[122]

$$\tilde{\Sigma}_1^s = \Sigma_1^s \equiv \bar{\Sigma}_1^s \tag{3.26}$$

$$\tilde{\Sigma}_2^s = \Sigma_2^s \equiv \bar{\Sigma}_2^s \tag{3.27}$$

For any spot market Nash equilibrium, equations (3.26) and (3.27) state that the optimal SF Σ_j^s will coincide with the imputed SF $\tilde{\Sigma}_j^s$, and we may define such an equilibrium optimal provisional spot market SF for firm j as $\bar{\Sigma}_j^s$. These equations must hold at all values of the arguments of $\tilde{\Sigma}_j^s$, so that we may also write Σ_j^s and $\bar{\Sigma}_j^s$ as $\Sigma_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)$ and $\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)$ respectively ($i,\ j=1,2;\ \ i\neq j$). If there exist multiple Nash equilibria in spot market SFs, we assume that firms successfully coordinate on a single equilibrium, denoted as $\bar{\Sigma}_j^s$ ($j=1,2$).[123]

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[122] We assume for now that such an equilibrium exists and examine later a simplified example (see chapter 5) for which we may prove existence.

[123] This is admittedly a strong assumption. We could appeal instead to refinements of Nash equilibrium such as "rationalizable strategies"; these are strategies that are best responses to beliefs that a

Replacing $\tilde{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)$ with $\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)$ in eq. (3.25) at this Nash equilibrium, we define firm 1's *equilibrium optimal provisional spot market profits*, $\bar{\pi}_i^{s*}$, as

$$\begin{aligned}\bar{\pi}_i^{s*}\left\{\bar{\Sigma}_j^s\left(\bullet;\tilde{q}_j^f,q_i^f\right),q_i^f,\varepsilon^s\right\}=\max_{p^s}\Big\{&p^s\cdot\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)\right]-p^s q_i^f\\&-C_i\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)\right]\Big\}.\end{aligned}\tag{3.28}$$

By our assumption, firms coordinate on a Nash equilibrium SF $\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)$. This function is no longer the arbitrary imputation $\tilde{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)$, but a specific function. We may thus re-express $\bar{\pi}_i^{s*}\left\{\bar{\Sigma}_j^s\left(\cdot;\tilde{q}_j^f,q_i^f\right),q_i^f,\varepsilon^s\right\}$ more succinctly as $\bar{\pi}_i^{s*}\left\{q_i^f,\tilde{q}_j^f,\varepsilon^s\right\}$,[124] and hence eq. (3.28) becomes

$$\bar{\pi}_i^{s*}\left\{q_i^f,\tilde{q}_j^f,\varepsilon^s\right\}=\max_{p^s}\bar{\pi}_i^s\left\{p^s,\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right),q_i^f,\varepsilon^s\right\}\tag{3.29}$$

where

$$\begin{aligned}\bar{\pi}_i^s\left\{p^s,\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right),q_i^f,\varepsilon^s\right\}=\;&p^s\cdot\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)\right]-p^s q_i^f\\&-C_i\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)\right].\end{aligned}\tag{3.30}$$

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firm might have about its rivals' strategies (Fudenberg and Tirole 1991, 49). These solution concepts tend to have little predictive power, however, and given the repeated interaction present in real-world electricity markets (not modeled here, as subsection 3.1.1 explains), the emergence of some degree of coordination on equilibria is certainly plausible.

In any event, in the simplified affine example that we solve in chapter 5, we will demonstrate the existence of a *unique* equilibrium in spot market SFs, so that the coordination problem among multiple equilibria does not arise.

[124] Redefining the arguments of $\bar{\pi}_i^{s*}$ (with a slight abuse of notation) and allowing the dependence of $\bar{\Sigma}_j^s$ on q_i^f to be incorporated into this redefined function $\bar{\pi}_i^{s*}$.

Let the *expected equilibrium optimal provisional spot market profits* for firm i be

$$\mathrm{E}\left(\bar{\pi}_i^{s*}\left\{q_i^f, \tilde{q}_j^f, \varepsilon^s\right\}\middle|\varepsilon_0^f\right), \tag{3.31}$$

where eqs. (3.29) and (3.30) give an expression for $\bar{\pi}_i^{s*}\left\{q_i^f, \tilde{q}_j^f, \varepsilon^s\right\}$, and the expectation in the expression (3.31) is taken with respect to ε^s, conditional on ε_0^f. The rationale for introducing this expectation is as follows. Firm i faces spot market uncertainty—embodied here in the demand shock ε^s—as it constructs its forward market bid in period 1. We assume that, being risk neutral, the firm accommodates this uncertainty via mathematical expectations as in the expression (3.31). After forward market uncertainty—represented here by ε_0^f—is revealed, firm i accommodates the remaining spot market uncertainty via its spot market SF bid, which is then ex post optimal for all realized values of ε^s given a forward market outcome ε_0^f.

Now let *total profits* for firm i in the multi-settlement SFE model, $\tilde{\pi}_i^{tot}$—given an imputed forward market quantity for firm j of $\tilde{q}_j^f$—be the sum of forward market revenue R_i^f and expected equilibrium optimal provisional spot market profits from the expression (3.31), that is,

$$\tilde{\pi}_i^{tot} = R_i^f + \mathrm{E}\left(\bar{\pi}_i^{s*}\left\{q_i^f, \tilde{q}_j^f, \varepsilon^s\right\}\middle|\varepsilon_0^f\right).$$

Using eq. (3.2), we may rewrite the above equation substituting $p^f q_i^f$ for R_i^f (and including the arguments of $\tilde{\pi}_i^{tot}$):

$$\tilde{\pi}_i^{tot}\left\{p^f, q_i^f, \tilde{q}_j^f, \varepsilon_0^f\right\} = p^f q_i^f + \mathrm{E}\left(\bar{\pi}_i^{s*}\left\{q_i^f, \tilde{q}_j^f, \varepsilon^s\right\}\middle|\varepsilon_0^f\right). \tag{3.32}$$

In eq. (3.32), firm i's forward market quantity, q_i^f, is equal to firm i's forward market residual demand function evaluated at p^f. The appropriate residual demand function to use here is that based on firm j's imputed admissible forward market SF, $\tilde{S}_j^f\left(p^f\right)$. Namely, we define[125]

$$q_i^f \equiv D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_j^f\left(p^f\right), \tag{3.33}$$

at an arbitrary ε_0^f. Substituting eq. (3.33) into eq. (3.32) for q_i^f and using $\tilde{S}_j^f\left(p^f\right)$ in place of $\tilde{q}_j^f$ as an argument in eq. (3.32) yields

$$\begin{aligned}
&\tilde{\pi}_i^{tot}\left\{p^f,\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_j^f\left(p^f\right)\right],\tilde{S}_j^f\left(p^f\right),\varepsilon_0^f\right\}\\
&\quad=p^f\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_j^f\left(p^f\right)\right]\\
&\quad\quad+\mathrm{E}\left(\bar{\pi}_i^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_j^f\left(p^f\right)\right],\tilde{S}_j^f\left(p^f\right),\varepsilon^s\right\}\middle|\varepsilon_0^f\right),
\end{aligned} \tag{3.34}$$

where eq. (3.29) gives an expression for $\bar{\pi}_i^{s*}\left\{q_i^f,\tilde{q}_j^f,\varepsilon^s\right\}$. Maximizing eq. (3.34) with respect to p^f will constitute firm i's *forward market objective*, given ε_0^f.

We now characterize firm i's forward market optimum given an imputed admissible forward market SF for firm j, $\tilde{S}_j^f\left(p^f\right)$. Eq. (3.34) gives an expression for firm i's total profits. For a subgame perfect Nash equilibrium in the forward market problem in any state of the world ε_0^f, a necessary condition will be that, given $\tilde{S}_j^f\left(p^f\right)$,

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125 In eq. (3.33), we refer to the forward market demand function $D^f\left(p^f,\varepsilon_0^f\right)$, which, though we introduced it in section 3.1.10, we have not yet defined explicitly. As noted in that section, $D^f\left(p^f,\varepsilon_0^f\right)$ is *endogenous* to the multi-settlement SFE model. Chapter 6 explains in detail how consumers' actions give rise to $D^f\left(p^f,\varepsilon_0^f\right)$, and also characterizes its properties.

firm i will choose an optimal—that is, "total profit-maximizing"—price $p^f = p_i^{f*}$ in the forward market.[126] Let the *optimal total profits* for firm i, $\tilde{\pi}_i^{tot*}$, be the maximized value of $\tilde{\pi}_i^{tot}$ at p_i^{f*}, that is,

$$\tilde{\pi}_i^{tot*}\left\{\tilde{S}_j^f(\bullet), \varepsilon_0^f\right\} = \max_{p^f} \tilde{\pi}_i^{tot}\left\{p^f, \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_j^f\left(p^f\right)\right], \tilde{S}_j^f\left(p^f\right), \varepsilon_0^f\right\}.$$

Substituting from eq. (3.34) for $\tilde{\pi}_i^{tot}\left\{p^f, \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_j^f\left(p^f\right)\right], \tilde{S}_j^f\left(p^f\right), \varepsilon_0^f\right\}$ in the above equation, we get

$$\begin{aligned}\tilde{\pi}_i^{tot*}\left\{\tilde{S}_j^f(\bullet), \varepsilon_0^f\right\} = \max_{p^f}\Big[& p^f\left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_j^f\left(p^f\right)\right] \\ & + \mathrm{E}\left(\bar{\pi}_i^{s*}\left\{\left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_j^f\left(p^f\right)\right], \tilde{S}_j^f\left(p^f\right), \varepsilon^s\right\} \middle| \varepsilon_0^f\right)\Big],\end{aligned} \tag{3.35}$$

where, recalling eqs. (3.29) and (3.30),

$$\bar{\pi}_i^{s*}\left\{q_i^f, \tilde{q}_j^f, \varepsilon^s\right\} = \max_{p^s} \bar{\pi}_i^s\left\{p^s, \bar{\Sigma}_j^s\left(p^s; \tilde{q}_j^f, q_i^f\right), q_i^f, \varepsilon^s\right\} \tag{3.36}$$

and where

$$\begin{aligned}\bar{\pi}_i^s\left\{p^s, \bar{\Sigma}_j^s\left(p^s; \tilde{q}_j^f, q_i^f\right), q_i^f, \varepsilon^s\right\} = p^s \cdot \left[D^s\left(p^s, \varepsilon^s\right) - \bar{\Sigma}_j^s\left(p^s; \tilde{q}_j^f, q_i^f\right)\right] - p^s q_i^f \\ - C_i\left[D^s\left(p^s, \varepsilon^s\right) - \bar{\Sigma}_j^s\left(p^s; \tilde{q}_j^f, q_i^f\right)\right].\end{aligned} \tag{3.37}$$

We defer consideration of equilibrium existence and uniqueness in the forward market subgame and hence of the existence and uniqueness of subgame perfect Nash

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[126] For now, we simply *assume* the existence of a unique equilibrium price p_i^{f*} for each ε_0^f; we consider this issue more formally in section 4.2.

equilibrium. Accordingly, we solve eqs. (3.35)–(3.37) given an arbitrary imputation, $\tilde{S}_j^f\left(p^f\right)$, for firm *j*.

Equations (3.35)–(3.37) comprise the forward market problem statement for firm *i*. Before discussing the solution strategy for this problem, we briefly review and summarize the foregoing derivation of these equations. As above, we start before the imposition of equilibrium in the spot market—namely, with eq. (3.22) for $\tilde{\pi}_i^s$—and review the steps involved in developing eqs. (3.35)–(3.37).

Examining the three additive terms in eq. (3.22), we see that the first term represents spot market revenue, the product of spot market price and the residual demand (given ε^s) met by firm *i* at that price. The second term is firm *i*'s contract settlement payment at the spot market price, p^s, with holders of forward contracts for q_i^f of output. The third term is the cost of production incurred by firm *i* for producing its spot market quantity, determined from the firm's residual demand function, given ε^s and evaluated at p^s. We maximize $\tilde{\pi}_i^s$ with respect to p^s to obtain $\tilde{\pi}_i^{s*}$, as on the left-hand side of eq. (3.25). Then, we impose a Nash equilibrium in spot market SFs $\bar{\Sigma}_j^s\left(p^s;\tilde{q}_j^f,q_i^f\right)$ in eq. (3.28), which yield profits $\bar{\pi}_i^{s*}$ as given by eq. (3.29). Next, eq. (3.34) takes the conditional expectation $\mathrm{E}\left(\bar{\pi}_i^{s*}\middle|\varepsilon_0^f\right)$, and computes $\tilde{\pi}_i^{tot}$ as the sum of forward market revenue—the product of forward market price and (given ε_0^f) the residual demand met by firm *i* at that price—and $\mathrm{E}\left(\bar{\pi}_i^{s*}\middle|\varepsilon_0^f\right)$. Finally, we maximize $\tilde{\pi}_i^{tot}$ with respect to p^f to yield $\tilde{\pi}_i^{tot*}$ on the left-hand side of eq. (3.35).

To conclude this section, we restate the SPNE (expression (3.1)) in light of the specification of $\bar{\Sigma}_i^s\left(p^s;\bullet\right)$ as $\bar{\Sigma}_i^s\left(p^s;\tilde{q}_i^f,q_j^f\right)$, as follows:

$$\left\{\bar{S}_i^f\left(p^f\right),\bar{\Sigma}_i^s\left(p^s;\tilde{q}_i^f,q_j^f\right)\right\},\quad i=1,2 \quad \Leftrightarrow \quad \begin{array}{c}\text{SPNE for the two-player,}\\ \text{multi-settlement market SFE game.}\end{array} \tag{3.38}$$

In the next section below, for concreteness, we rewrite eqs. (3.35)–(3.37) for firm $i=1$, explain why the backward induction solution algorithm is appropriate, and show how it gives rise to firm 1's optimal SF, $S_1^f\left(p^f\right)$. Then, in chapter 4, we solve firm 1's forward market problem.

3.4 Solving firm 1's forward market problem via backward induction

Firm 1's forward market problem in the multi-settlement market setting is to maximize its total profits, $\tilde{\pi}_1^{tot}$, given $\tilde{S}_2^f\left(p^f\right)$ for firm 2, in any state of the world ε_0^f. We denote such maximized profits as $\tilde{\pi}_1^{tot*}$, given by eqs. (3.35)–(3.37), rewritten below for $i=1$ and $j=2$:

$$\begin{aligned}\tilde{\pi}_1^{tot*}\left\{\tilde{S}_2^f\left(\bullet\right),\varepsilon_0^f\right\}=\max_{p^f}\Big[&p^f\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\\&+\mathrm{E}\left(\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}\middle|\varepsilon_0^f\right)\Big],\end{aligned} \tag{3.39}$$

where

$$\bar{\pi}_1^{s*}\left\{q_1^f,\tilde{q}_2^f,\varepsilon^s\right\}=\max_{p^s}\bar{\pi}_1^s\left\{p^s,\bar{\Sigma}_2^s\left(p^s;\tilde{q}_2^f,q_1^f\right),q_1^f,\varepsilon^s\right\} \tag{3.40}$$

and

$$\bar{\pi}_1^s\left\{p^s, \bar{\Sigma}_2^s\left(p^s;\tilde{q}_2^f, q_1^f\right), q_1^f, \varepsilon^s\right\} = p^s \cdot \left[D^s\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^s\left(p^s;\tilde{q}_2^f, q_1^f\right)\right] - p^s q_1^f \\ - C_1\left[D^s\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^s\left(p^s;\tilde{q}_2^f, q_1^f\right)\right], \tag{3.41}$$

and $\bar{\Sigma}_2^s\left(p^s;\tilde{q}_2^f, q_1^f\right)$ in eqs. (3.40) and (3.41) is firm 2's equilibrium optimal provisional spot market SF.[127] Although not immediately evident from eqs. (3.39)–(3.41), firm 1's decision variables in period 1 are its forward market supply quantities for all feasible prices p^f; the locus of such points, at an optimum, is the firm's optimal SF, $S_1^f\left(p^f\right)$. Since $S_1^f\left(p^f\right)$ does not appear explicitly in the above equations, it is useful to describe how this problem formulation, in fact, ultimately yields a function $S_1^f\left(p^f\right)$. This is the goal of this section.

Note first that the relationships

$$\tilde{q}_2^f = \tilde{S}_2^f\left(p^f\right) \tag{3.42}$$

and

$$q_1^f = \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right] \tag{3.43}$$

are reflected implicitly in eqs. (3.40) and (3.41). Equation (3.42) is due to the definition of firm 2's imputed admissible forward market SF (see subsection 3.1.5). Equation (3.43) is from the market-clearing condition: if p^f is a market-clearing price for the forward market, then firm 1's forward market quantity, q_1^f, must be equal to the firm's

[127] As we demonstrate in chapter 5 below, the equilibrium SF $\bar{\Sigma}_2^s$ is a function only of exogenous parameters that are common knowledge, so that firm 1 may compute $\bar{\Sigma}_2^s$ in the course of solving its forward market problem.

residual demand function evaluated at p^f. Given an imputation $\tilde{S}_2^f\left(p^f\right)$ and for any ε_0^f, the forward quantities $\tilde{q}_2^f$ and q_1^f are functions of p^f from eqs. (3.42) and (3.43). Firm 1 computes from eqs. (3.39)–(3.41) (for the assumed ε_0^f) its optimal price $p^f = p_1^{f*}\left(\varepsilon_0^f\right)$, the argmax for its forward market problem. Subsection 3.4.2 describes how, by repeating this computation of $p^f = p_1^{f*}\left(\varepsilon_0^f\right)$ pointwise for all possible ε_0^f, firm 1 may construct its optimal SF, $S_1^f\left(p^f\right)$.[128]

The sequential structure of firm 1's forward market problem suggests *backward induction* as the appropriate solution algorithm. Indeed, as described above, the first backward induction step begins by solving for firms' optimal provisional *spot market* SFs (parameterized in terms of the realized forward market quantities, $\hat{q}_1^f$ and $\hat{q}_2^f$). Then, we impose Nash equilibrium in the spot market, yielding equilibrium spot market SFs. Next, in the second backward induction step, we construct firms' optimal admissible *forward market* SFs, given the equilibrium spot market result from the first step. The following two subsections describe these two backward induction steps in more detail.

3.4.1 First stage: The spot market

Consider first the spot market.[129] Here, assuming a realization of the forward market demand shock $\hat{\varepsilon}_0^f$ and realized forward market quantities $\hat{q}_1^f$ and $\hat{q}_2^f$, firm 1's spot

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[128] Naturally, we will also assume that firm 2 computes its optimal SF, $S_2^f\left(p^f\right)$ in an analogous fashion. We then impose equilibrium in the forward market, given the assumed spot market Nash equilibrium. The resulting strategies (i.e., the sequence of forward and spot market actions) for each firm constitute a subgame perfect Nash equilibrium for the multi-settlement SFE model.

[129] This subsection follows closely the presentation of Klemperer and Meyer (1989, 1251).

market residual demand at a price p^s is the difference between total demand in the spot market and the quantity that firm 2 is willing to supply there at that price. Thus if firm 2 is committed to a (strictly increasing) imputed provisional SF $\tilde{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$ $=\overline{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$, firm 1's spot market residual demand function is $D^s\left(p^s,\varepsilon^s\right)$ $-\overline{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$.

Following KM, since ε^s is a scalar, the set of points along firm 1's spot market residual demand functions satisfying the first-order condition (FOC) corresponding to eqs. (3.40) and (3.41) (fixing $q_1^f=\hat{q}_1^f$ and $\tilde{q}_2^f=\hat{q}_2^f$), as ε^s varies over all its possible values, is a one-dimensional function in p^s-q^s space. If this function can be described by an admissible SF $q_1^s\left(\hat{q}_1^f,\hat{q}_2^f\right)\equiv\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ that intersects each realization of firm 1's spot market residual demand function once and only once, then by committing to Σ_1^s, firm 1 can achieve ex post optimal adjustment to the shock ε^s. In this case, Σ_1^s is firm 1's unique optimal provisional SF for the spot market in response to $\overline{\Sigma}_2^s$.

Firm 2 may also solve its version of the spot market problem, which we obtain from eqs. (3.39)–(3.41) by interchanging subscripts "$_1$" and "$_2$" throughout these equations. Firm 2 solves its problem in the same manner as did firm 1, described above, given the imputed provisional spot market SF for firm 1, $\overline{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$. Firm 2 obtains Σ_2^s as its unique optimal provisional SF for the spot market in response to $\overline{\Sigma}_1^s$. Our earlier assumption that each firm's imputed and optimal provisional spot market SFs coincide at each p^s, $\hat{q}_1^f$, and $\hat{q}_2^f$ satisfies the Nash equilibrium condition for the spot market; we denoted the equilibrium SFs as $\bar{\Sigma}_i^s$.

For now, we *assume* that the set of points yielding equilibrium optimal provisional spot market profits $\bar{\pi}_1^{s*}$ for firm 1 (see eqs. (3.40) and (3.41)) can be described by the provisional SF Σ_1^s—and likewise for firm 2—and investigate later whether, under our hypotheses, there exist equilibria in which this is indeed the case.

3.4.2 *Second stage: The forward market*

In the second stage of firm 1's backward induction algorithm, we move back in time to period 1, before the forward market clears and before revelation of the uncertain demand shock ε_0^f.[130] Accordingly, we revert to the notation for as-yet-unknown values of ε_0^f and quantities q_1^f and $\tilde{q}_2^f$ (to indicate this, we write these parameters now without carets and use firm 2's imputed forward market quantity, $\tilde{q}_2^f$).

Consider the expression for firm 1's residual demand in the first term of eq. (3.39)'s objective function. Analogous to the situation in the spot market, firm 1's forward market residual demand at any price p^f is the difference between total demand in the forward market and the quantity that firm 2 is willing to supply there at that price. Thus if firm 2 is committed to a (strictly increasing) imputed admissible SF $\tilde{S}_2^f\left(p^f\right)$, firm 1's forward market residual demand function is $D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)$.

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[130] Note 129 applies here, as well.

Since ε_0^f is a scalar, the set of points satisfying the FOC corresponding to eq. (3.39) for maximum total profits (given $\tilde{S}_2^f\left(p^f\right)$), as ε_0^f varies over all its possible values, is a one-dimensional function in p^f-q^f space. If this function can be described by an admissible SF $q_1^f \equiv S_1^f\left(p^f\right)$ that intersects each realization of firm 1's forward market residual demand function once and only once, then by committing to S_1^f, firm 1 can achieve ex post optimal (in the sense of eqs. (3.39)–(3.41)) adjustment to the shock ε_0^f. In this case, S_1^f is firm 1's unique optimal admissible SF for the forward market in response to $\tilde{S}_2^f$.

Firm 2 may also solve its version of the forward market problem, which we obtain from eq. (3.39)–(3.41) by interchanging subscripts "$_1$" and "$_2$" throughout these equations. Firm 2 solves its problem in the same manner as did firm 1, described above, given the imputed admissible forward market SF for firm 1, $\tilde{S}_1^f\left(p^f\right)$. Firm 2 obtains S_2^f as its unique optimal admissible SF for the forward market in response to $\tilde{S}_1^f$. At this point, we impose the Nash equilibrium condition for the forward market, which is that $\tilde{S}_i^f$ and S_i^f coincide at each p^f for $i = 1, 2$; we denote this equilibrium SF as $\overline{S}_i^f$.

For now, we *assume* that the set of points yielding maximum total profits for firm 1 given $\tilde{S}_2^f\left(p^f\right)$ (see eq. (3.39) for $\tilde{\pi}_1^{tot*}$) can be described by the admissible SF S_1^f—and likewise for firm 2—and investigate later whether, under our hypotheses, there exist equilibria in which this is indeed the case.

3.4.3 Discussion

Each version of the first stage of the backward induction problem (the spot market—see subsection 3.4.1) assumes a *fixed* value of ε_0^f. This stage is nested within the problem's second stage (the forward market—see subsection 3.4.2), in which we construct S_1^f in pointwise fashion by solving the overall problem repeatedly for *all feasible* ε_0^f, given $\tilde{S}_2^f$.

This nested, hierarchical structure yields a forward market SF for firm 1 that maximizes its total profits $\tilde{\pi}_1^{tot}$ for all feasible ε_0^f. Eq. (3.39) defines $\tilde{\pi}_1^{tot*}$ in terms of the expected value of equilibrium optimal provisional spot market profits. To highlight the distinct contributions of the spot and forward markets to $\tilde{\pi}_1^{tot*}$, we could say that S_1^f will yield firm 1's *ex post optimal total profits,* $\tilde{\pi}_1^{tot*}$*, assuming ex ante expected equilibrium optimal provisional spot market profits,* $\mathrm{E}\left(\bar{\pi}_1^{s*}\middle|\varepsilon_0^f\right)$. This is the notion of optimality exhibited by forward market SFs in this thesis. The firm's actual (i.e., ex post optimal) spot market profits will be determined by spot market SF bidding in period 2.

The relationship between the optimal provisional spot market SFs and the optimal admissible spot market SFs should now be clear. The optimal provisional spot market SFs, $\Sigma_1^s\left(p^s;q_1^f,q_2^f\right)$ and $\Sigma_2^s\left(p^s;q_2^f,q_1^f\right)$, are functions of the form $\Sigma_i^s:\mathbb{R}^3\to\mathbb{R}$ since the forward quantities are still unknown when constructing forward market bids. Once these values of q_1^f and q_2^f have been revealed (as $q_1^f=\tilde{q}_1^f=\hat{q}_1^f$ and $q_2^f=\tilde{q}_2^f=\hat{q}_2^f$, say) in period 2, each firm may take these values $\hat{q}_1^f$ and $\hat{q}_2^f$ into account in constructing and submitting its optimal admissible spot market SF which, as market rules stipulate, have the form $S_1^s\left(p^s\right)$ and $S_2^s\left(p^s\right)$. These admissible SFs are functions of the form $S_i^s:\mathbb{R}\to\mathbb{R}$ (that is, they lie in the p^s-q_i^s plane); by construction, they are also the projections (fixing firms' forward quantities at $\hat{q}_1^f$ and $\hat{q}_2^f$) of $\Sigma_1^s\left(p^s;q_1^f,q_2^f\right)$ and $\Sigma_2^s\left(p^s;q_2^f,q_1^f\right)$ onto these planes. Algebraically, the relationship between these two types of spot market SF is, for realized $\hat{q}_i^f$ and $\hat{q}_j^f$,

$$S_i^s\left(p^s\right)=\Sigma_i^s\left(p^s;\hat{q}_i^f,\hat{q}_j^f\right)\forall\ p^s\ \ (i,j=1,2;\ i\neq j). \tag{3.44}$$

In this sense, then, the optimal provisional spot market SFs $\Sigma_i^s\left(p^s;\hat{q}_i^f,\hat{q}_j^f\right)$ are consistent with the optimal admissible spot market SFs $S_i^s\left(p^s\right)$, reflecting subsection 3.1.1's assumption of a closed-loop information structure.

Philosophy is perfectly right in saying that life must be understood backward. But then one forgets the other clause—that it must be lived forward.

—Kierkegaard, *Journals and Papers*

Sell when you can; you are not for all markets.

—Shakespeare, *As You Like It*

4 Derivation of the optimal forward market SF

THIS CHAPTER derives firm 1's optimal forward market SF using the backward induction procedure sketched in section 3.4 above. Accordingly, section 4.1 below analyzes the spot market in the first stage of the problem. Section 4.2 is then devoted to the forward market in the second stage of the problem. This chapter follows closely the presentation of Klemperer and Meyer (1989, 1251-2).

4.1 First stage: The spot market

We begin by recasting the expression for firm 1's equilibrium optimal provisional spot market profits, $\bar{\pi}_1^{s*}$ (eq. (3.40)). We solve this equation given a realized, arbitrary forward market shock $\varepsilon_0^f = \hat{\varepsilon}_0^f$ and forward quantities $q_1^f = \hat{q}_1^f$ and $\tilde{q}_2^f = \hat{q}_2^f$ for firms 1 and 2, respectively, and given an assumed (though not yet realized), arbitrary value of ε^s. As noted in subsection 3.4.1, we also assume that firm 2 is committed to a (strictly increasing) imputed provisional SF $\tilde{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)=\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$. Firm 1's spot market residual demand function is then $D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$.

Accordingly, firm 1's provisional spot market profit maximization problem becomes

$$\bar{\pi}_1^{s*}\left\{\hat{q}_1^f,\hat{q}_2^f,\varepsilon^s\right\}=\max_{p^s}\bar{\pi}_1^s\left\{p^s,\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\} \tag{4.1}$$

and

$$\begin{aligned}\bar{\pi}_1^s\left\{p^s,\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}=p^s\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]-p^s\hat{q}_1^f\\ -C_1\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right].\end{aligned} \tag{4.2}$$

The FOC of eq. (4.1) with respect to p^s (assuming an interior solution) is

$$\begin{aligned}&\frac{d\bar{\pi}_1^s\left\{p^s,\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}}{dp^s}\\ &=\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]-\hat{q}_1^f\\ &\quad+\left\{p^s-C_1'\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\right\}\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\\ &=0,\end{aligned} \tag{4.3}$$

where primes on spot market demand and the SFs denote derivatives with respect to p^s.

If the objective function in eq. (4.1) is globally strictly concave in p^s (Appendix B verifies the second-order condition), then eq. (4.3) implicitly determines, given $\hat{q}_1^f$ and

$\hat{q}_2^f$, firm 1's unique provisional spot market profit-maximizing price, $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$, for the assumed value of ε^s. The corresponding profit-maximizing quantity is

$$D^s\left(p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right),\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right);\hat{q}_2^f,\hat{q}_1^f\right)\equiv q_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right).$$

The functions $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$ and $q_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$ represent in parameterized form firm 1's set of ex post optimal points in the spot market (given $\hat{q}_1^f$ and $\hat{q}_2^f$) as the firm's spot market residual demand function shifts. If $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$ is partially invertible[131] with respect to ε^s, this locus can be written as a function of spot market price to quantity as

$$q_1^s=\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\equiv q_1^{s*}\left(\left(p_1^{s*}\right)_{\varepsilon^s}^{-1}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right);\hat{q}_1^f,\hat{q}_2^f\right), \tag{4.4}$$

where $\left(p_1^{s*}\right)_{\varepsilon^s}^{-1}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ denotes the partial inverse of $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$ with respect to ε^s. Since $\partial D^s\left(p^s,\varepsilon^s\right)/\partial\varepsilon^s>0$, no two realizations of firm 1's residual demand function can intersect; this condition, together with uniqueness of $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$ for each ε^s implies that $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ intersects firm 1's residual demand function once and only once for each ε^s, at $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$.[132] Hence, $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ is firm 1's optimal provisional spot market SF in response to firm 2's imputed provisional spot market SF, $\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$.

---

[131] We demonstrate partial invertibility in the context of a simplified affine example below (see section 5.4).

[132] See Appendix A for a proof of these claims.

Let us rewrite eq. (4.3) so that it implicitly defines the function $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$. First, however, we follow Klemperer and Meyer (1989, 1250) and invert the spot market demand function with respect to ε^s, noting that this inverse exists since $\partial D^s\left(p^s,\varepsilon^s\right)/\partial\varepsilon^s>0$. Let

$$e^s\left(Q^s,p^s\right)$$

denote the value of the shock ε^s for which total spot market demand is Q^s at price p^s, that is, $e^s\left(Q^s,p^s\right)$ satisfies $Q^s=D^s\left(p^s,e^s\left(Q^s,p^s\right)\right)$. To make explicit the relationship between ε^s and the firms' forward market positions $\hat{q}_1^f$ and $\hat{q}_2^f$, we first write the spot market-clearing condition—given $\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$ and, from eq. (4.4), $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$—as[133]

$$\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)+\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)=Q^s. \tag{4.5}$$

Hence, from the definition of the function $e^s\left(Q^s,p^s\right)$ and eq. (4.5), we have

$$\varepsilon^s=e^s\left[\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)+\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),p^s\right]. \tag{4.6}$$

Now, in eq. (4.3), replace

$$q_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)\equiv D^s\left(p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right),\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right);\hat{q}_2^f,\hat{q}_1^f\right) \tag{4.7}$$

---

[133] Note that eq. (4.5) represents the spot market-clearing condition as firm 1 would conceive it, in terms of the SF that it imputes to firm 2, $\bar{\Sigma}_2^s$, and its own *optimal* SF, Σ_1^s. Firm 2's conception of the spot market-clearing condition would be symmetric to eq. (4.5), and in any Nash equilibrium, these two conceptions of the spot market-clearing condition will coincide.

by $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ and use eq. (4.6) for ε^s to replace $D^{s'}\left(p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right),\varepsilon^s\right)$ by $D^{s'}\left(p^s,e^s\left[\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)+\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),p^s\right]\right)$ so that eq. (4.3) becomes

$$
\begin{aligned}
&\frac{d\bar{\pi}_1^s\left\{p^s,\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}}{dp^s} \\
&\qquad=\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)-\hat{q}_1^f \\
&\qquad\quad+\left[p^s-C_1'\left[\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\right] \\
&\qquad\quad\cdot\left[D^{s'}\left(p^s,e^s\left[\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)+\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),p^s\right]\right)\right. \\
&\qquad\qquad\left.-\bar{\Sigma}_2^{s'}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right] \\
&\qquad=0.
\end{aligned} \tag{4.8}
$$

We assumed earlier in eq. (3.6) that $\partial^2 D^s\left(p^s,\varepsilon^s\right)/\partial p^s\partial\varepsilon^s=0$, that is, the shock ε^s translates the spot market demand function horizontally. We may therefore rewrite the term $D^{s'}\left(p^s,e^s\left[\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)+\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),p^s\right]\right)$ in eq. (4.8) simply as $D_0^{s'}\left(p^s\right)$, recalling eq. (3.7). Doing this and rearranging eq. (4.8), we have for firm 1 the implicit differential equation

$$\left[\bar{\Sigma}_2^{s'}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)-D_0^{s'}\left(p^s\right)\right]\left\{p^s-C_1'\left[\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\right\}=\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)-\hat{q}_1^f. \tag{4.9}$$

Note that we could solve firm 2's problem to obtain a result completely symmetric to eq. (4.9) with firms' subscripts 1 and 2 interchanged.

The necessary Nash equilibrium condition in either stage game is that each firm's optimal SF is identical to the SF that its rival imputes to it. Given that each firm's SF satisfies its optimality conditions (e.g., eq. (4.9) for firm 1 and likewise for firm 2), the Nash equilibrium condition becomes a necessary and sufficient condition for a (pure strategy) Nash equilibrium in SFs. In the present derivation of the spot market's provisional solution in which each firm imputes to its rival the rival's optimal SF, this Nash equilibrium condition is

$$\Sigma_i^s\left(p^s;\hat{q}_i^f,\hat{q}_j^f\right) \equiv \bar{\Sigma}_i^s\left(p^s;\hat{q}_i^f,\hat{q}_j^f\right) \quad (i,\, j=1,2;\; i \neq j)\,, \tag{4.10}$$

where we have defined $\bar{\Sigma}_i^s\left(p^s;\hat{q}_i^f,\hat{q}_j^f\right)$ (see subsection 3.1.5) as firm i's equilibrium optimal provisional spot market SF. Impose this Nash equilibrium condition by recasting eq. (4.9) in terms of these equilibrium SFs.[134] That is, for each of the two firms, substitute into eq. (4.9) from eq. (4.10) letting, for firm 1, $i=1$ and $j=2$,

$$\left[\bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)-D_0^{s\prime}\left(p^s\right)\right]\left\{p^s-C_1'\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\right\}=\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)-\hat{q}_1^f \tag{4.11}$$

and, for firm 2, $i=2$ and $j=1$,

$$\left[\bar{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)-D_0^{s\prime}\left(p^s\right)\right]\left\{p^s-C_2'\left[\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\right\}=\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)-\hat{q}_2^f\,. \tag{4.12}$$

We call eq. (4.11) the *equilibrium optimality condition* for firm 1's equilibrium optimal provisional spot market SF $\bar{\Sigma}_1^s$, implicitly defining this function (and similarly for eq. (4.12) and $\bar{\Sigma}_2^s$ for firm 2). Finally, recall our assumption (see note 90) that if there are

[134] For arbitrary values ε^s, $\hat{q}_1^f$, and $\hat{q}_2^f$, the firms' respective optimal spot market price functions $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$ and $p_2^{s*}\left(\varepsilon^s;\hat{q}_2^f,\hat{q}_1^f\right)$ must coincide in any spot market Nash equilibrium, that is, $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)=p_2^{s*}\left(\varepsilon^s;\hat{q}_2^f,\hat{q}_1^f\right)\equiv p^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$. We assumed in section 3.1—and will prove in section 5.4 for a simplified affine example—that $p_1^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$ (and hence also $p^{s*}\left(\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$) is invertible.

multiple Nash equilibria in the spot market subgame, firms successfully coordinate on a particular spot market equilibrium to be anticipated.

For purposes of comparison with previous work, we make the temporary assumption that the price-cost margins $p^s - C_i'\left[\bar{\Sigma}_i^s\left(p^s;\hat{q}_i^f,\hat{q}_j^f\right)\right]$ are nonzero. This allows us to rearrange eqs. (4.11) and (4.12) as

$$\bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)=\frac{\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)-\hat{q}_1^f}{p^s-C_1'\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]}+D_0^{s\prime}\left(p^s\right), \tag{4.13}$$

and for firm 2,

$$\bar{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)=\frac{\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)-\hat{q}_2^f}{p^s-C_2'\left[\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]}+D_0^{s\prime}\left(p^s\right). \tag{4.14}$$

Comparing eq. (4.13) with Klemperer and Meyer's (1989, 1252) optimality condition for the symmetric single-market SFE, namely,

$$S'(p)=\frac{S(p)}{p-C'\left(S(p)\right)}+D'(p), \tag{4.15}$$

we see that, with the exceptions of the arguments $\hat{q}_1^f$ and $\hat{q}_2^f$ in eq. (4.13) and the assumption of symmetric firms (with symmetric costs) that underlies eq. (4.15), the structure of the two equations is identical. We have already argued that we may treat the higher-dimensional SFs in eq. (4.13) as two-dimensional projections in the p^s-q^s plane, since for any particular iteration of eq. (4.1), the arguments $\hat{q}_1^f$ and $\hat{q}_2^f$ are fixed. Thus, the functions $\bar{\Sigma}_1^s$ and $\bar{\Sigma}_2^s$ in eq. (4.13) are closely analogous to the supply function S in eq. (4.15). We could view KM's optimality condition (rewritten above as eq. (4.15)), therefore, as simply a special case of eq. (4.13) in which $\hat{q}_1^f = \hat{q}_2^f = 0$ and firms are symmetric. We will see later when solving explicitly a simplified version of eq. (4.13) that the SF solutions of the two equations are indeed closely related.

This completes the first backward induction stage to find the provisional solution for the spot market. In the second stage considered in the next section, we seek the solution to firm 1's forward market problem.

4.2 Second stage: The forward market

In confronting the second stage of the backward induction problem for firm 1, we move back in time to period 1, before the forward market clears and before revelation of the forward market parameters ε_0^f , q_1^f , and $\tilde{q}_2^f$. Accordingly, we revert to the notation for the not-yet-revealed values of these parameters and write them now without carets. We first recast the forward market problem by replacing the arguments q_1^f and $\tilde{q}_2^f$ in eqs. (3.40) and (3.41) with functions of p^f using eqs. (3.43) and (3.42). Then, we solve this problem given an assumed (though not yet realized) arbitrary value of ε_0^f .

With these substitutions, eqs. (3.39)–(3.41) become

$$\begin{aligned} \tilde{\pi}_1^{tot*}\left\{\tilde{S}_2^f\left(\bullet\right),\varepsilon_0^f\right\} = \max_{p^f}\Big[& p^f\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right] \\ & +\mathrm{E}\left(\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}\middle|\varepsilon_0^f\right)\Big], \end{aligned} \tag{4.16}$$

where

$$
\begin{aligned}
&\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\} \\
&\qquad =\max_{p^s}\bar{\pi}_1^s\Big\{p^s,\bar{\Sigma}_2^s\Big\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\Big\}, \\
&\qquad\qquad \left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\varepsilon^s\Big\}
\end{aligned}
\tag{4.17}
$$

and

$$
\begin{aligned}
&\bar{\pi}_1^s\Big\{p^s,\bar{\Sigma}_2^s\Big\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\Big\},\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\varepsilon^s\Big\} \\
&\quad =p^s\cdot\Big(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\Big\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\Big\}\Big) \\
&\qquad -p^s\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right] \\
&\qquad -C_1\Big(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\Big\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\Big\}\Big).
\end{aligned}
\tag{4.18}
$$

Together, eqs. (4.16)–(4.18) constitute firm 1's forward market optimization problem: maximize total expected profits—given a value of the forward market demand shock, ε_0^f, and a (strictly increasing) imputed admissible forward market SF for firm 2, $\tilde{S}_2^f\left(p^f\right)$—by choosing p^f.

The FOC of eqs. (4.16)–(4.18) with respect to p^f—denoting[135] the objective function of eq. (4.16) as $\tilde{\pi}_1^{tot}\left\{p^f,\tilde{S}_2^f\left(p^f\right),\varepsilon_0^f\right\}$ and assuming an interior solution—is

[135] With a slight abuse of notation since we had earlier defined $\tilde{\pi}_i^{tot}=\tilde{\pi}_i^{tot}\left\{p^f,q_i^f,\tilde{q}_j^f,\varepsilon_0^f\right\}$ (see eq. (3.32)) as a function of four arguments.

$$\frac{d\tilde{\pi}_1^{tot}\left\{p^f,\tilde{S}_2^f\left(p^f\right),\varepsilon_0^f\right\}}{dp^f}=\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]+p^f\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]$$
$$+\mathrm{E}\left[\left.\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right|\varepsilon_0^f\right] \quad (4.19)$$
$$=0,$$

where the primes on forward market demand and SFs denote derivatives with respect to p^f. We may evaluate the derivative inside the expectation in eq. (4.19) by first applying the chain rule to the left-hand side of eq. (4.17):[136]

$$\left.\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right|\varepsilon_0^f$$
$$=\left(\frac{\partial\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial q_1^f}\cdot\frac{dq_1^f}{dp^f}\right. \quad (4.20)$$
$$\left.\left.+\frac{\partial\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial\tilde{q}_2^f}\cdot\frac{d\tilde{q}_2^f}{dp^f}\right)\right|\varepsilon_0^f,$$

where we have used the fact that $d\varepsilon^s/dp^f=0$ since (as we will see in chapter 6) ε^s depends only on Period 2 (spot market) uncertainty once ε_0^f is fixed.

To evaluate the partial derivatives of $\bar{\pi}_1^{s*}$ with respect to q_1^f and $\tilde{q}_2^f$ in eq. (4.20), we apply the envelope theorem to eq. (4.17). This yields

---

[136] Recalling eqs. (3.43) and (3.42), we see that the first and second arguments of $\bar{\pi}_1^{s*}$ are q_1^f and $\tilde{q}_2^f$, respectively. It will be useful shorthand in eq. (4.20) above to define derivatives of $\bar{\pi}_1^{s*}$ with respect to these forward market quantities.

$$\frac{\partial\overline{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial q_1^f}$$
$$= \partial\overline{\pi}_1^s\left\{p^s,\overline{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\},\right. \qquad (4.21)$$
$$\left.\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\varepsilon^s\right\}\Big/\partial q_1^f$$

and

$$\frac{\partial\overline{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial \tilde{q}_2^f}$$
$$= \partial\overline{\pi}_1^s\left\{p^s,\overline{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\},\right. \qquad (4.22)$$
$$\left.\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\varepsilon^s\right\}\Big/\partial q_2^f .$$

Suppressing the arguments of $\overline{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\}$ as $\overline{\Sigma}_2^s\{\cdots\}$, for brevity, the right-hand sides of eqs. (4.21) and (4.22) become, respectively (using eq. (4.18)),

$$\partial\overline{\pi}_1^s\left\{p^s,\overline{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\},\right.$$
$$\left.\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\varepsilon^s\right\}\Big/\partial q_1^f \qquad (4.23)$$
$$= p^s\left(-\frac{\partial\overline{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}\right)-p^s-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^s\{\cdots\}\right)\left(-\frac{\partial\overline{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}\right)$$

and

$$\partial\overline{\pi}_1^s\left\{p^s,\overline{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\},\right.$$
$$\left.\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\varepsilon^s\right\}\Big/\partial q_2^f \qquad (4.24)$$
$$= p^s\left(-\frac{\partial\overline{\Sigma}_2^s\{\cdots\}}{\partial \tilde{q}_2^f}\right)-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^s\{\cdots\}\right)\left(-\frac{\partial\overline{\Sigma}_2^s\{\cdots\}}{\partial \tilde{q}_2^f}\right).$$

Combining eqs. (4.21) and (4.23) and simplifying, we get

$$\begin{aligned}&\frac{\partial \bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial q_1^f}\\&\quad=-\left[p^s-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\right]\cdot\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}-p^s.\end{aligned}\tag{4.25}$$

Combining eqs. (4.22) and (4.24) and simplifying, we get

$$\begin{aligned}&\frac{\partial \bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial \tilde{q}_2^f}\\&\quad=-\left[p^s-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\right]\cdot\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial \tilde{q}_2^f}.\end{aligned}\tag{4.26}$$

Recalling eqs. (4.4)–(4.7) above and the associated discussion, now that we have differentiated, we may replace the argument of the marginal cost functions in eqs. (4.25) and (4.26), $D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}$, with $\bar{\Sigma}_1^s\left\{p^s;\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right)\right\}$ $\equiv\bar{\Sigma}_1^s\{\cdots\}$. Doing this, eqs. (4.25) and (4.26) become

$$\begin{aligned}&\frac{\partial \bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial q_1^f}\\&\quad=-\left[p^s-C_1'\left(\bar{\Sigma}_1^s\{\cdots\}\right)\right]\cdot\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}-p^s;\end{aligned}\tag{4.27}$$

and

$$\begin{aligned}&\frac{\partial \bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial \tilde{q}_2^f}\\&\quad=-\left[p^s-C_1'\left(\bar{\Sigma}_1^s\{\cdots\}\right)\right]\cdot\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial \tilde{q}_2^f}.\end{aligned}\tag{4.28}$$

We may substitute into eq. (4.20) from eqs. (4.27) and (4.28) to obtain

$$\left.\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right|\varepsilon_0^f$$
$$=-\left.\left\{\left[p^s-C_1'\left(\bar{\Sigma}_1^s\{\cdots\}\right)\right]\cdot\left[\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}\cdot\frac{dq_1^f}{dp^f}+\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial\tilde{q}_2^f}\cdot\frac{d\tilde{q}_2^f}{dp^f}\right]+p^s\cdot\frac{dq_1^f}{dp^f}\right\}\right|\varepsilon_0^f. \tag{4.29}$$

Note that we may interpret the second bracketed term on the right-hand side of eq. (4.29) as

$$\left[\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}\cdot\frac{dq_1^f}{dp^f}+\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial\tilde{q}_2^f}\cdot\frac{d\tilde{q}_2^f}{dp^f}\right]=\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial p^f}, \tag{4.30}$$

where the partial derivative $\partial\bar{\Sigma}_2^s\{\cdots\}/\partial p^f$ holds p^s constant. Again using eqs. (3.43) and (3.42), we may express the derivatives dq_1^f/dp^f and $d\tilde{q}_2^f/dp^f$ in eq. (4.29) as:

$$\frac{dq_1^f}{dp^f}=\frac{d}{dp^f}\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]=D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right); \tag{4.31}$$

and

$$\frac{d\tilde{q}_2^f}{dp^f}=\frac{d\tilde{S}_2^f\left(p^f\right)}{dp^f}=\tilde{S}_2^{f\prime}\left(p^f\right). \tag{4.32}$$

Finally, we substitute eqs. (4.31) and (4.32) into eq. (4.29) and the result, in turn, into the forward market FOC (eq. (4.19)) to obtain

$$
\begin{aligned}
&\frac{d\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right), \varepsilon_0^f\right)}{dp^f} \\
&\quad = \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right] + p^f\left[D^{f\prime}\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\qquad - \mathrm{E}\Bigg(\Bigg\{\Big[p^s - C_1'\left(\bar{\Sigma}_1^s\{\cdots\}\right)\Big] \\
&\qquad\qquad \cdot\left[\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}\cdot\left[D^{f\prime}\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] + \frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial \tilde{q}_2^f}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\qquad\qquad + p^s\left[D^{f\prime}\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right]\Bigg\}\Bigg|\varepsilon_0^f\Bigg) \\
&\quad = 0.
\end{aligned}
$$

Given ε_0^f, the functions $D^{f\prime}\left(p^f, \varepsilon_0^f\right)$ and $\tilde{S}_2^{f\prime}(\cdot)$ are both constant as ε^s varies. Hence, the slope of residual demand $\left[D^{f\prime}\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right]$ inside the expectation operator is itself constant with respect to ε^s (though the expectation does act upon p^s, which premultiplies this term). Therefore, this term denoting the slope of residual demand may be treated as a constant in the above equation, and taken outside of the expectation. Using this fact and again writing the arguments of $\bar{\Sigma}_1^s\{\cdots\}$ and $\bar{\Sigma}_2^s\{\cdots\}$ explicitly, this FOC becomes

$$
\begin{aligned}
&\frac{d\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right), \varepsilon_0^f\right)}{dp^f} \\
&\quad = \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right] + \left[p^f - \mathrm{E}\left(p^s \middle| \varepsilon_0^f\right)\right]\left[D^{f\prime}\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\qquad - \mathrm{E}\Bigg(\Bigg\{\Bigg[p^s - C_1'\left(\bar{\Sigma}_1^s\left\{p^s; \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right], \tilde{S}_2^f\left(p^f\right)\right\}\right)\Bigg] \\
&\qquad\qquad \cdot \Bigg[\frac{\partial \bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right]\right\}}{\partial q_1^f} \\
&\qquad\qquad\quad \cdot \left[D^{f\prime}\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\qquad\qquad\quad + \frac{\partial \bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), \left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right]\right\}}{\partial \tilde{q}_2^f} \cdot \tilde{S}_2^{f\prime}\left(p^f\right)\Bigg]\Bigg\}\Bigg| \varepsilon_0^f\Bigg) \\
&\quad = 0.
\end{aligned}
\tag{4.33}
$$

If the objective function in eq. (4.16) is globally strictly concave in p^f (Appendix B gives sufficient conditions for the second-order condition to hold), then eq. (4.33) implicitly determines firm 1's unique profit-maximizing price, $p_1^{f*}\left(\varepsilon_0^f\right)$, for each value of ε_0^f. The corresponding profit-maximizing quantity is

$$D^f\left(p_1^{f*}\left(\varepsilon_0^f\right), \varepsilon_0^f\right) - \tilde{S}_2^f\left(p_1^{f*}\left(\varepsilon_0^f\right)\right) \equiv q_1^{f*}\left(\varepsilon_0^f\right).$$

The functions $p_1^{f*}\left(\varepsilon_0^f\right)$ and $q_1^{f*}\left(\varepsilon_0^f\right)$ represent in parameterized form firm 1's set of ex post optimal points in the forward market as the firm's forward market residual demand function shifts. If $p_1^{f*}\left(\varepsilon_0^f\right)$ is invertible,[137] this locus can be written as a function of forward market price to quantity:

---

[137] We demonstrate the invertibility of $p_1^{f*}\left(\bullet\right)$ in section 5.4 below.

$$q_1^f = S_1^f\left(p^f\right) \equiv q_1^{f*}\left(\left(p_1^{f*}\right)^{-1}\left(p^f\right)\right), \tag{4.34}$$

where $\left(p_1^{f*}\right)^{-1}\left(p^f\right)$ denotes the inverse of $p_1^{f*}\left(\varepsilon_0^f\right)$. Since $\partial D^f\left(p^f,\varepsilon_0^f\right)/\partial\varepsilon_0^f > 0$, no two realizations of firm 1's residual demand function can intersect; this condition, together with uniqueness of $p_1^{f*}\left(\varepsilon_0^f\right)$ for each ε_0^f implies that $S_1^f\left(p^f\right)$ intersects firm 1's residual demand function once and only once for each ε_0^f, at $p_1^{f*}\left(\varepsilon_0^f\right)$.[138] Hence $S_1^f\left(p^f\right)$ is firm 1's optimal admissible forward market SF in response to firm 2's imputed admissible forward market SF, $\tilde{S}_2^f\left(p^f\right)$.

Let us rewrite eq. (4.33) so that it implicitly defines the function $S_1^f\left(p^f\right)$. First, however, we follow Klemperer and Meyer (1989, 1250) and invert the forward market demand function with respect to ε_0^f, noting that this inverse exists since $\partial D^f\left(p^f,\varepsilon_0^f\right)/\partial\varepsilon_0^f > 0$. Let

$$e^f\left(Q^f, p^f\right)$$

denote the value of the shock ε_0^f for which total forward market demand is Q^f at price p^f, that is, $e^f\left(Q^f, p^f\right)$ satisfies $Q^f = D^f\left(p^f, e^f\left(Q^f, p^f\right)\right)$. Now, in eq. (4.33), replace

$$q_1^{f*}\left(\varepsilon_0^f\right) \equiv D^f\left(p_1^{f*}\left(\varepsilon_0^f\right),\varepsilon_0^f\right) - \tilde{S}_2^f\left(p_1^{f*}\left(\varepsilon_0^f\right)\right) \tag{4.35}$$

[138] See Appendix A for a proof of these claims.

by $S_1^f\left(p^f\right)$, and use $e^f\left(Q^f, p^f\right)$ as defined above with $Q^f = S_1^f\left(p^f\right) + \tilde{S}_2^f\left(p^f\right)$[139] to replace $D^{f\prime}\left(p_1^{f*}\left(\varepsilon_0^f\right), \varepsilon_0^f\right)$ by $D^{f\prime}\left(p^f, e^f\left[S_1^f\left(p^f\right) + \tilde{S}_2^f\left(p^f\right), p^f\right]\right)$. Then, the FOC (4.33) becomes

$$\begin{aligned}
&\frac{d\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right), \varepsilon_0^f\right)}{dp^f} \\
&\quad = S_1^f\left(p^f\right) \\
&\qquad + \left[p^f - \mathrm{E}\left(p^s \middle| \varepsilon_0^f\right)\right]\left[D^{f\prime}\left(p^f, e^f\left[S_1^f\left(p^f\right) + \tilde{S}_2^f\left(p^f\right), p^f\right]\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\qquad - \mathrm{E}\Bigg(\Bigg\{\left[p^s - C_1'\left(\bar{\Sigma}_1^s\left\{p^s; S_1^f\left(p^f\right), \tilde{S}_2^f\left(p^f\right)\right\}\right)\right] \\
&\qquad\qquad \cdot\Bigg[\frac{\partial\bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), S_1^f\left(p^f\right)\right\}}{\partial q_1^f} \\
&\qquad\qquad\qquad \cdot\left[D^{f\prime}\left(p^f, e^f\left[S_1^f\left(p^f\right) + \tilde{S}_2^f\left(p^f\right), p^f\right]\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\qquad\qquad\qquad + \frac{\partial\bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), S_1^f\left(p^f\right)\right\}}{\partial\tilde{q}_2^f} \cdot \tilde{S}_2^{f\prime}\left(p^f\right)\Bigg]\Bigg\}\Bigg| \varepsilon_0^f\Bigg) \\
&\quad = 0.
\end{aligned} \tag{4.36}$$

We assumed earlier in eq. (3.12) that $\partial^2 D^f\left(p^f, \varepsilon_0^f\right) / \partial p^f \partial\varepsilon_0^f = 0$, that is, the shock ε_0^f translates the forward market demand function horizontally. We may therefore write $D^{f\prime}\left(p^f, e^f\left[S_1^f\left(p^f\right) + \tilde{S}_2^f\left(p^f\right), p^f\right]\right)$ simply as $D_0^{f\prime}\left(p^f\right)$, recalling eq. (3.13). Making this change in eq. (4.36) yields

[139] This is the forward market clearing condition as firm 1 would conceive it. The argument of note 133 applies here, mutatis mutandis.

$$
\begin{aligned}
\frac{d\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right)\right)}{dp^f} &= S_1^f\left(p^f\right) + \left[p^f - \mathrm{E}\left(p^s \middle| p^f\right)\right]\left[D_0^{f\prime}\left(p^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\quad - \mathrm{E}\Bigg(\Bigg\{\Bigg[p^s - C_1'\left(\bar{\Sigma}_1^s\left\{p^s; S_1^f\left(p^f\right), \tilde{S}_2^f\left(p^f\right)\right\}\right)\Bigg] \\
&\qquad \cdot \Bigg[\frac{\partial \bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), S_1^f\left(p^f\right)\right\}}{\partial q_1^f} \\
&\qquad \cdot \left[D_0^{f\prime}\left(p^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] \\
&\qquad + \frac{\partial \bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), S_1^f\left(p^f\right)\right\}}{\partial \tilde{q}_2^f} \cdot \tilde{S}_2^{f\prime}\left(p^f\right)\Bigg]\Bigg\} \Bigg| p^f\Bigg) \\
&= 0,
\end{aligned} \tag{4.37}
$$

where we now condition expectations in eq. (4.37) on firm 1's *optimal price* $p^f = p_1^{f*}\left(\varepsilon_0^f\right)$, thus suppressing explicit dependence of the FOC on ε_0^f.[140] Finally, we may rearrange eq. (4.37) as

$$
\left[\tilde{S}_2^{f\prime}\left(p^f\right) - D_0^{f\prime}\left(p^f\right)\right]\left[p^f - \mathrm{E}\left(p^s \middle| p^f\right)\right] = S_1^f\left(p^f\right) + \psi_1\left(p^f\right), \tag{4.38}
$$

where we define $\psi_1\left(p^f\right)$ as

[140] Note that we may condition in eq. (4.37) on either p^f or ε_0^f under our assumption (justified in section 5.4) that $p_1^{f*}(\bullet)$ is invertible, and hence p^f and ε_0^f are one-to-one. We also commit a slight abuse of notation in eq. (4.37) in expressing $\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right)\right)$ as a function of only two rather than three arguments ($\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right), \varepsilon_0^f\right)$), as in eq. (4.36) and the foregoing analysis.

$$\psi_1\left(p^f\right) \equiv -\mathrm{E}\left(\left\{\left[p^s - C_1'\left(\bar{\Sigma}_1^s\left\{p^s; S_1^f\left(p^f\right), \tilde{S}_2^f\left(p^f\right)\right\}\right)\right]\right.\right.$$

$$\cdot\left[\frac{\partial\bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), S_1^f\left(p^f\right)\right\}}{\partial q_1^f}\cdot\left[D_0^{f\prime}\left(p^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right]\right. \tag{4.39}$$

$$\left.\left.\left.+\frac{\partial\bar{\Sigma}_2^s\left\{p^s; \tilde{S}_2^f\left(p^f\right), S_1^f\left(p^f\right)\right\}}{\partial \tilde{q}_2^f}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right]\right\}\middle| p^f\right).$$

Note that we could solve firm 2's forward market problem to obtain a result completely symmetric to eqs. (4.38) and (4.39), but with firms' subscripts 1 and 2 interchanged.

The necessary Nash equilibrium condition in either stage game is that each firm's optimal SF is identical to the SF that its rival imputes to it. Given that each firm's SF satisfies its optimality conditions (i.e., eqs. (4.38) and (4.39) for firm 1 in the forward market and likewise for firm 2), the Nash equilibrium condition becomes a necessary and sufficient condition for a Nash equilibrium in SFs. In the present derivation of the forward market's solution, this Nash equilibrium condition is

$$\tilde{S}_i^f\left(p^f\right) = S_i^f\left(p^f\right) \equiv \bar{S}_i^f\left(p^f\right) \quad (i, j = 1, 2;\ i \neq j), \tag{4.40}$$

where we have defined $\bar{S}_i^f\left(p^f\right)$ (see subsection 3.1.5) as firm *i*'s equilibrium optimal admissible forward market SF. Impose this Nash equilibrium condition by recasting eqs. (4.38) and (4.39) in terms of these equilibrium SFs, that is, substitute into these equations from eq. (4.40) letting $i = 1$ and $j = 2$, yielding[141]

---

[141] We may make an argument analogous to that in note 134 above that for an arbitrary value ε_0^f, the firms' respective optimal forward market price functions $p_1^{f*}\left(\varepsilon_0^f\right)$ and $p_2^{f*}\left(\varepsilon_0^f\right)$ must coincide in any

$$\left[\bar{S}_2^{f\prime}\left(p^f\right)-D_0^{f\prime}\left(p^f\right)\right]\left[p^f-\mathrm{E}\left(p^s\middle|p^f\right)\right]=\bar{S}_1^f\left(p^f\right)+\psi_1\left(p^f\right), \tag{4.41}$$

where we redefine $\psi_1\left(p^f\right)$ as

$$\begin{aligned}\psi_1\left(p^f\right)\equiv-\mathrm{E}\Bigg(\Bigg\{\Big[p^s-C_1'\Big(\bar{\Sigma}_1^s\left\{p^s;\bar{S}_1^f\left(p^f\right),\bar{S}_2^f\left(p^f\right)\right\}\Big)\Big]\\ \cdot\Bigg[\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\bar{S}_2^f\left(p^f\right),\bar{S}_1^f\left(p^f\right)\right\}}{\partial\bar{q}_1^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\bar{S}_2^{f\prime}\left(p^f\right)\right]\\ +\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\bar{S}_2^f\left(p^f\right),\bar{S}_1^f\left(p^f\right)\right\}}{\partial\bar{q}_2^f}\cdot\bar{S}_2^{f\prime}\left(p^f\right)\Bigg]\Bigg\}\Bigg|p^f\Bigg).\end{aligned}$$

Replacing $\bar{S}_i^f\left(p^f\right)$ with firm i's equilibrium forward market quantity $\bar{q}_i^f$, the above expression becomes

$$\begin{aligned}\psi_1\left(p^f\right)\equiv-\mathrm{E}\Bigg(\Bigg\{\Big[p^s-C_1'\Big(\bar{\Sigma}_1^s\left\{p^s;\bar{q}_1^f,\bar{q}_2^f\right\}\Big)\Big]\\ \cdot\Bigg[\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\bar{q}_2^f,\bar{q}_1^f\right\}}{\partial\bar{q}_1^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\bar{S}_2^{f\prime}\left(p^f\right)\right]\\ +\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\bar{q}_2^f,\bar{q}_1^f\right\}}{\partial\bar{q}_2^f}\cdot\bar{S}_2^{f\prime}\left(p^f\right)\Bigg]\Bigg\}\Bigg|p^f\Bigg).\end{aligned} \tag{4.42}$$

We say that eqs. (4.41) and (4.42) constitute the *forward market equilibrium optimality condition* for firm 1's equilibrium optimal admissible forward market SF

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forward market Nash equilibrium, i.e., $p_1^{f*}\left(\varepsilon_0^f\right)=p_2^{f*}\left(\varepsilon_0^f\right)\equiv p^{f*}\left(\varepsilon_0^f\right)$. We assumed in section 3.1—and will prove in section 5.4—that $p_1^{f*}\left(\varepsilon_0^f\right)$ (and hence also $p^{f*}\left(\varepsilon_0^f\right)$) is invertible.

$\overline{S}_1^f\left(p^f\right)$.[142] Comparing the structures of eqs. (4.41) and (4.42) with that of Klemperer and Meyer's (1989, 1252) optimality condition for the single-market SFE—given above as eq. (4.15)—we see that they differ in three respects:

1. As with the spot market solution for firm 1 (eq. (4.13)), we derived eqs. (4.41) and (4.42) for two asymmetric firms with asymmetric cost functions. Eq. (4.15) (from KM), in contrast, assumed two symmetric firms.
2. In eq. (4.41), the expected spot price $\mathrm{E}\left(p^s \middle| p^f\right)$ plays the role of marginal cost $C'\left(S\right)$ in eq. (4.15). This structural similarity suggests that we may interpret the expected spot price as a *marginal opportunity cost* to a (risk-neutral) supplier of a particular quantity contracted in the forward market.
3. Equation (4.41) contains the term $\psi_1\left(p^f\right)$ (see eq. (4.42)), whereby KM's optimality condition, eq. (4.15), has no such term. Appendix C provides an economic interpretation of $\psi_1\left(p^f\right)$. Namely, $\psi_1\left(p^f\right)$ is the expected change in firm 1's equilibrium optimal provisional spot profits caused by a marginal change in p^f while netting out the expected change in its forward contract settlement payment, $\left(-p^s q_1^f\right)$, due to this change in p^f. In other words, $\psi_1\left(p^f\right)$ captures the expected effect of a marginal change in p^f on firm 1's spot market revenue

142 An analogous procedure would yield the corresponding equilibrium optimality condition for firm 2's equilibrium optimal admissible forward market SF $\overline{S}_2^f\left(p^f\right)$.

less production cost.[143] We may express this interpretation of $\psi_1\left(p^f\right)$ algebraically as

$$\psi_1\left(p^f\right)=\mathrm{E}\left(\left.\frac{d\bar{\pi}_1^{s*}\left\{\bar{q}_1^f,\bar{q}_2^f,\varepsilon^s\right\}}{dp^f}\right|p^f\right)+\frac{d\bar{q}_1^f}{dp^f}\cdot\mathrm{E}\left(p^s\middle|p^f\right), \tag{4.43}$$

as Appendix C demonstrates.[144] The merit of this result is that the relationship of the optimality condition, eqs. (4.41) and (4.42), to the original problem statement, eqs. (4.16)–(4.18), is then particularly transparent. Later in chapter 8, we also identify $\psi_1\left(p^f\right)$ as firm 1's *strategic effect*, accounting, in part, for the firm's participation in the forward market.

Equations (4.11) and (4.41) (using (4.42))—and the analogous equations for firm 2—constitute a mixed system of differential equations: partial differential equations in $\bar{\Sigma}_i^s\left\{p^s;\bar{q}_i^f,\bar{q}_j^f\right\}$ and total differential equations in $\bar{S}_i^f\left(p^f\right)$, with the cross-equation restrictions of $\bar{q}_i^f=\bar{S}_i^f\left(p^f\right)$, $i,j=1,2;i\neq j$. From these systems, we observe that the forward and spot markets are coupled in at least two ways:

1. In general, firms' equilibrium forward market quantities $\hat{q}_i^f=\bar{q}_i^f$ and $\hat{q}_j^f=\bar{q}_j^f$ enter both firms' provisional spot market SFs $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ as arguments.

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[143] Recall that the definition of equilibrium optimal provisional spot market profits for firm *i* from eqs. (3.29) and (3.30) included not only the firm's spot market revenue less production cost, but also the forward contract settlement payment, in this case $\left(-p^s q_1^f\right)$.

[144] In eq. (4.43), we have denoted the change in firm 1's forward market quantity for a change in p^f as $d\bar{q}_1^f/dp^f=D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)=D_0^{f\prime}\left(p^f\right)-\bar{S}_2^{f\prime}\left(p^f\right)$, the slope of firm 1's forward market residual demand function, recalling eqs. (3.43), (3.13), and (4.40).

2. Both the level function and the various partial derivatives of firm *j*'s provisional spot market SF $\bar{\Sigma}_j^s\left(p^s;\bar{q}_j^f,\bar{q}_i^f\right)$, enter the function $\psi_i\left(p^f\right)$, which itself appears in firm *i*'s forward market equilibrium optimality condition in section 4.2.

In addition, we make explicit a third relationship between the two markets in chapter 5, where we establish how the equilibrium spot market price p^s depends on firms' equilibrium forward market quantities $\bar{q}_1^f$ and $\bar{q}_2^f$.

Solutions to the aforementioned mixed system of differential equations would be difficult to characterize in the general case. Newbery (1998, 733) anticipated this complexity, noting the "double infinity of solutions" that arises when we permit a continuum of spot market equilibria (characterized by eqs. (4.11) and (4.12)) for every forward market equilibrium, themselves elements in a continuum. The continuum of solutions in each market exists because each solution corresponds to a particular initial condition (or boundary condition) in a continuum of such conditions for each differential equation.[145] In the following chapter, we appeal to several simplifying assumptions that render eqs. (4.41) and (4.42) more tractable.

[145] These continua of initial conditions might arise, for example, due to physical capacity constraints or limits on financial contracting related to credit risk.

> Far better an approximate answer to the right question, which is often vague, than the exact answer to the wrong question, which can always be made precise.
>
> —J.W. Tukey, The Future of Data Analysis

> Everything should be made as simple as possible, but not simpler.
>
> —Einstein

5 A simplified affine example

THIS CHAPTER introduces an affine example that simplifies the spot market—and ultimately, also the forward market—analysis. Section 5.1 below begins by introducing three assumptions regarding affine functional forms in the spot market, and section 5.2 explores the implications of these assumptions for the spot market SFs. Section 5.3 conducts comparative statics analysis for spot market SFs with respect to cost and demand function parameters. Next, we investigate the implications of the affine functional form assumptions for optimal spot market prices and the forward market optimality conditions in sections 5.4 and 5.5, respectively. Section 5.6 concludes.

5.1 Affine functional forms

We now invoke several simplifying assumptions in order to carry the analysis further. From this point forward, let us restrict ourselves to the case in which the following three assumptions hold concerning the spot market:

AFFINE SPOT MARKET DEMAND FUNCTION: *The spot market demand function is affine, having the form* $D^s\left(p^s, \varepsilon^s\right) = -\gamma^s p^s + \varepsilon^s$. *Thus, the spot market demand function's slope* $D^{s\prime}\left(p^s, \varepsilon^s\right) \equiv \partial D^s\left(p^s, \varepsilon^s\right) \big/ \partial p^s$ *is* $-\gamma^s$, *where* $\gamma^s > 0$.[146]

AFFINE MARGINAL PRODUCTION COST FUNCTIONS: *Each firm has a* quadratic *production cost function* $C_i\left(q_i^s\right)$, *given by*

$$C_i\left(q_i^s\right) = c_{0i} q_i^s + \tfrac{1}{2} c_i q_i^{s2}, \quad q_i^s \geq 0,$$

where $c_{0i} \geq 0$ and $c_i > 0$ ($i = 1, 2$).

Marginal production cost $C_i'\left(q_i^s\right)$ *for each firm is then also affine*:

$$C_i'\left(q_i^s\right) = c_{0i} + c_i q_i^s, \quad q_i^s \geq 0. \tag{5.1}$$

AFFINE SPOT MARKET SFS (EQUILIBRIUM SELECTION): *The provisional spot market SFs* $\overline{\Sigma}_i^s\left(p^s; \overline{q}_i^f, \overline{q}_j^f\right)$ $\left(i, j = 1, 2;\ i \neq j\right)$ *are* affine *in* p^s. *That is,* $\overline{\Sigma}_i^s\left(p^s; \overline{q}_i^f, \overline{q}_j^f\right)$ *is of the form*

$$\overline{\Sigma}_i^s\left(p^s; \overline{q}_i^f, \overline{q}_j^f\right) = \alpha_i^s + \beta_i^s p^s \quad \left(i, j = 1, 2;\ i \neq j\right), \tag{5.2}$$

[146] In subsection 6.4.4, we relate the magnitude of γ^s to parameters of consumers' utility functions.

where α_i^s *is the quantity axis intercept and* β_i^s *the slope of the (affine) projection of* $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ *onto the* p^s*-*q^s *plane.*

A principal goal of this chapter is to investigate the effects that these various simplifying assumptions have on the spot market supply functions, the optimal spot market price function, and the forward market equilibrium optimality conditions. In section 5.3, we also perform comparative statics analysis for the spot market in this affine case.

While Affine Spot Market SFs may at first appear to be a fairly strong assumption, there are two theoretical grounds for selecting affine spot market SFs for further study. First, given the cost functions, the affine spot market SF is the limiting equilibrium action as the range of uncertainty in spot market demand increases. Second, stability arguments favor the selection of the affine SF over alternative strictly concave or strictly convex SFs. We elucidate these arguments below. Finally, apart from these theoretical justifications, the affine functional form in the spot market simplifies the analysis.

Klemperer and Meyer (1989, 1261 (Proposition 4)) show in their single-market SFE analysis that when the support of the stochastic demand shock is bounded above, there exists a continuum, or *connected set*, of SFEs consisting of both strictly convex and strictly concave SFs, as well as an affine SF in the interior of the set (we call this the *equilibrium set*).[147] As we increase the upper endpoint $\hat{\varepsilon}$ of this support, the continuum of equilibrium SFs narrows as the most concave and most convex SFs drop out of the

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[147] In addition, KM require that demand and marginal cost functions be linear for sufficiently large price and quantity. While their proof also assumes *symmetric* firms, this property does not appear to be necessary for their result (see Rudkevich 1999).

equilibrium set.[148] In the limit as $\widehat{\varepsilon} \to \infty$, considering the sequence of equilibrium sets associated with each value of $\widehat{\varepsilon}$, this sequence converges to an equilibrium set having a single element, the affine SF. For this reason, KM conclude that "[f]or unbounded support, there exists a unique SFE and it is linear" [or more generally, affine, given affine marginal cost functions with strictly positive intercepts]. It is straightforward to show that this argument based on the single-market SFE carries over to the spot market, as well, in the multi-settlement market context.[149] We do not make the rather strong assumption here that $\widehat{\varepsilon}^s$ is necessarily unbounded. Rather, we simply restrict ourselves under the Affine Spot Market SFs assumption to the class of affine spot market supply functions, noting that this assumption becomes less restrictive the larger is $\widehat{\varepsilon}^s$.

More recent work on the stability of SFE models has shown that under plausible conditions, (single-market) non-affine SFs are *unstable*, as elaborated below. In particular, under assumptions analogous to the Affine Spot Market Demand Function and Affine Marginal Production Cost Functions assumptions above,[150] Baldick and Hogan (2001, 30 (Theorem 6)) find that single-market SFEs comprising either (1) strictly concave SFs for each firm or (2) strictly convex SFs for each firm are "unstable."[151]

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[148] This is because at extreme values of the demand shock ε, the SFs having the greatest curvature violate the second-order condition for profit maximization beyond a certain point in their domain.

[149] Whether this argument also holds in the *forward market* within the multi-settlement market, however, is a matter for further research; see chapter 7.

[150] And assuming, in addition (as we did in subsection 3.1.8), that suppliers face no binding capacity constraints.

[151] See Baldick and Hogan (2001, 30) for details. These authors define an unstable SFE in the following intuitive sense: An SFE is *unstable* when small perturbations to equilibrium SFs elicit best responses from firms that deviate further from this equilibrium (with respect to an appropriate norm on the function space of SFs) than do the originally assumed perturbations. The authors do not address the case of SFEs in which the concavity of the equilibrium SFs varies across firms (e.g., when some firms have strictly

Based on their analysis, it is reasonable to conjecture (although we do not prove this here) that Baldick and Hogan's aforementioned result for the single-market setting will carry over (at least to the spot market) in the multi-settlement market environment. Therefore, if stability of the equilibrium is a salient—and desirable—characteristic, the affine spot market SFs studied here are also those most of interest on stability grounds.

Apart from stability considerations, Green (1996) has made the case that affine spot market SFs may be reasonable approximations to the actual equilibrium SFs, particularly at certain demand levels.[152] In addition, the Affine Spot Market SFs assumption is naturally attractive, as it makes the multi-settlement SFE model more tractable analytically. Finally, this assumption also facilitates comparisons with previous work (e.g., Green 1999a), which has similarly focused, for the most part, on the affine or linear cases.

5.2 Implications for the spot market supply functions

This subsection solves for the parameters α_i^s and explains how to solve for the β_i^s (see eq. (5.2)).[153] For concreteness, we conduct the analysis for firm 1. Begin by substituting from eq. (5.2) for each firm's affine spot market SF into eq. (4.13), firm 1's spot market

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concave SFs while others' SFs are strictly convex). Whether such cases arise is not known, but if they do, their stability properties are unknown.

[152] For the case of a single market with marginal cost passing through the origin, Green notes (p. 209, n. 3) that the slope at the origin of all SFs through this point is equal to the slope of the unique linear SF also passing through the origin. Thus, any affine SF approximates an arbitrary (nonlinear) equilibrium SF at low demand levels. It may be shown that an analogous result holds for spot market SFs in a multi-settlement market, whereby the approximation is valid in the neighborhood of the point on the marginal cost function at the forward contract quantity (see, e.g., Figure 5.1).

[153] The results of this subsection are consistent with those of Green (1999a), who examined a forward contract market using conjectural variations interacting with a spot market using SFE.

equilibrium optimality condition. Doing so (and imposing Nash equilibrium in the forward market) yields, for all market-clearing p^s,

$$\left(\beta_2^s+\gamma^s\right)\left\{p^s-\left[c_{01}+c_1\left(\alpha_1^s+\beta_1^s p^s\right)\right]\right\}=\alpha_1^s+\beta_1^s p^s-\overline{q}_1^f . \tag{5.3}$$

Simplifying and collecting factors of p^s and constant terms, we get

$$\begin{aligned}&\beta_2^s\left(1-c_1\beta_1^s\right)p^s-\beta_2^s\left(c_{01}+c_1\alpha_1^s\right)\\&\qquad=\left[\beta_1^s\left(1+c_1\gamma^s\right)-\gamma^s\right]p^s+\left[\alpha_1^s\left(1+c_1\gamma^s\right)+c_{01}\gamma^s-\overline{q}_1^f\right].\end{aligned}$$

For this equation to hold for any market-clearing price p^s, the factors of p^s on either side of this equation must be equal, as must the constant terms. Equating these terms, defining the dimensionless parameter ϕ_i as

$$\phi_i\equiv\frac{1}{1+c_i\left(\gamma^s+\beta_j^s\right)}\quad (i,\, j=1,2;\; i\neq j)\,, \tag{5.4}$$

and solving for α_1^s and for β_1^s (in terms of β_2^s), we have[154]

$$\alpha_1^s=\phi_1\overline{q}_1^f-c_{01}\beta_1^s \tag{5.5}$$

and

$$\beta_1^s=\phi_1\left(\gamma^s+\beta_2^s\right). \tag{5.6}$$

Considering the (equilibrium) forward market positions $\overline{q}_1^f$ and $\overline{q}_2^f$, we see that α_1^s in eq. (5.5) depends only on $\overline{q}_1^f$ and not on $\overline{q}_2^f$, while β_1^s does not depend on either firm's

[154] Equations (5.5) and (5.6) are consistent with Green's (1999a) eqs. (7) for the duopoly case that he studies.

forward market position. The interpretation is that the affine spot market SF depends only on one's own quantity awarded in the forward market and not on the competitor's quantity. This observation is an instance of Green's (1999a) finding concerning the effect of forward contract positions in his linear SF model and the distinction between quantities and stage game actions in the SFE setting. As Green noted, "[firm *j*'s] *quantity* is decreasing in [firm *i*'s] contract sales, but its [optimal spot market action]—its *supply function*—is not affected by them."[155] Our assumption of affine SFs is critical to this property, however; non-affine spot market SFs do depend on the rival's forward market quantity.

Rewriting eq. (5.6) for generic firms *i* and *j* and using eq. (5.4), we find that the parameter ϕ_i may also be written as

$$\phi_i = 1 - c_i \beta_i^s, \quad i = 1, 2. \tag{5.7}$$

Note that ϕ_i is a function only of spot market constants, and assuming that $\beta_j^s > 0$, eq. (5.4) implies, further, that

$$0 < \phi_i < 1. \tag{5.8}$$

Using eq. (5.5), we may rewrite eq. (5.2) for firm 1's spot SF in terms of β_1^s as[156]

$$\overline{\Sigma}_1^s \left(p^s; \overline{q}_1^f, \overline{q}_2^f \right) = \left(\phi_1 \overline{q}_1^f - c_{01} \beta_1^s \right) + \beta_1^s p^s. \tag{5.9}$$

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155 Green (1999a, 116) (emphasis in original). An increase in a firm's forward market position ("contract sales," in Green's parlance) decreases its rival's quantity by depressing the equilibrium spot market price, thereby calling forth less supply from its rival, given the rival's fixed spot market SF. On the nature of this effect in the present model, see section 5.4 below.

156 For consistency, we maintain $\overline{q}_2^f$ as an argument of $\overline{\Sigma}_1^s$, although we note that $\overline{q}_2^f$ does not appear on the right-hand side of eq. (5.9), as explained above.

We may write analogous expressions that characterize firm 2's spot SF by interchanging subscripts 1 and 2 in eqs. (5.5), (5.6), and (5.9):

$$\alpha_2^s = \phi_2 \bar{q}_2^f - c_{02}\beta_2^s , \tag{5.10}$$

$$\beta_2^s = \phi_2\left(\gamma^s + \beta_1^s\right), \tag{5.11}$$

and

$$\bar{\Sigma}_2^s\left(p^s;\bar{q}_2^f,\bar{q}_1^f\right) = \left(\phi_2\bar{q}_2^f - c_{02}\beta_2^s\right) + \beta_2^s p^s . \tag{5.12}$$

The equations (5.9) and (5.12) for the firms' spot market SFs indicate that we may interpret the dimensionless parameter ϕ_i introduced in eq. (5.4) as the partial derivative of firm i's SF $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ with respect to i's forward market quantity, that is,

$$\frac{\partial\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)}{\partial\bar{q}_i^f} = \phi_i > 0, \quad i, j = 1, 2;\ i \neq j . \tag{5.13}$$

In other words, we may construe ϕ_i as the sensitivity, at the margin, of firm i's spot market quantity bid (at a given price p^s) to changes in its forward market quantity $\bar{q}_i^f$.

5.3 Comparative statics for the spot market

When solved simultaneously, eqs. (5.6) and (5.11) yield a quadratic form in β_1^s and β_2^s, the slopes of the respective firms' affine spot market SFs, such that $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$ ($i, j = 1, 2;\ i \neq j$). This system of β_i^s is a special case (for $n = 2$) of the general n-firm model studied by Rudkevich (1999), in which firms with affine marginal costs bid affine SFs into a centrally-cleared market. For the duopoly case studied here, Rudkevich's result implies that the quadratic form in β_1^s and β_2^s has exactly one root in which both β_1^s and β_2^s are positive. Thus, there is a *unique* solution $\left(\beta_1^s, \beta_2^s\right)$ corresponding to a strictly increasing spot market SF for each firm.[157] Given that $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$, we also have from the definition of ϕ_i in eq. (5.4) that $\phi_i = \phi_i\left(c_i, c_j, \gamma^s\right)$.

Table 5.1 below reports the signs of the partial derivatives of $\beta_i^s\left(c_i, c_j, \gamma^s\right)$ and $\phi_i\left(c_i, c_j, \gamma^s\right)$ as derived in Appendix D.1 via differentiation of eqs. (5.6) and (5.11), as well as of definition (5.4) for ϕ_i.

TABLE 5.1: COMPARATIVE STATICS OF $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$ AND $\phi_i = \phi_i\left(c_i, c_j, \gamma^s\right)$ WITH RESPECT TO THE PARAMETERS c_i, c_j, AND γ^s $(i, j = 1, 2;\ i \neq j)$ (SEE APPENDIX D.1 FOR DETAILS)

$\frac{\partial \beta_i^s}{\partial c_i} < 0$	$\frac{\partial \phi_i}{\partial c_i} < 0$
$\frac{\partial \beta_i^s}{\partial c_j} < 0$	$\frac{\partial \phi_i}{\partial c_j} > 0$
$\frac{\partial \beta_i^s}{\partial \gamma^s} > 0$	$\frac{\partial \phi_i}{\partial \gamma^s} < 0$

The signs of the partial derivatives given in Table 5.1 are invariant with respect to the parameter values c_i, c_j, and γ^s. The comparative statics effects in the table for β_i^s indicate that—as intuition might suggest—as either firm's marginal cost function or the spot market demand function becomes steeper, the spot market SF slopes β_i^s become

[157] The exact analytical expression for $\beta_i^s\left(c_i, c_j, \gamma^s\right)$ is straightforward but tedious to obtain from eqs. (5.6) and (5.11); we do not require it for the present analysis and so do not solve for it explicitly here.

steeper. Moreover, again using eqs. (5.4), (5.6), and (5.11), we may show that the following inequalities obtain at all parameter values for the derivatives of β_i^s and β_j^s with respect to c_i and c_j ($i, j = 1, 2;\ i \neq j$; see Appendix D.2 for details):

$$\left|\frac{\partial \beta_i^s}{\partial c_i}\right| > \left|\frac{\partial \beta_i^s}{\partial c_j}\right|, \tag{5.14}$$

and

$$\left|\frac{\partial \beta_i^s}{\partial c_i}\right| > \left|\frac{\partial \beta_j^s}{\partial c_i}\right|. \tag{5.15}$$

Inequalities (5.14) and (5.15) indicate that the effect of changing firm *i*'s own marginal cost function slope c_i on the slope β_i^s of *i*'s spot market SF $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ is greater in magnitude than either

1. the effect on β_i^s when changing the corresponding parameter c_j for *i*'s rival, *j* (eq. (5.14)),[158] or
2. the effect of changing c_i on the slope β_j^s of *j*'s spot market SF $\bar{\Sigma}_j^s\left(p^s;\bar{q}_j^f,\bar{q}_i^f\right)$ (eq. (5.15)).

The general insight here—consistent with intuition—is that a version of diagonal dominance holds for a Jacobian matrix of derivatives of the form

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[158] Recall from Table 5.1 that both of these effects are negative.

$$\begin{bmatrix} \left|\frac{\partial \beta_1^s}{\partial c_1}\right| & \cdots & \left|\frac{\partial \beta_1^s}{\partial c_n}\right| \\ \vdots & \ddots & \vdots \\ \left|\frac{\partial \beta_n^s}{\partial c_1}\right| & \cdots & \left|\frac{\partial \beta_n^s}{\partial c_n}\right| \end{bmatrix}, \tag{5.16}$$

in which each diagonal element of the matrix (5.16) is larger than the off-diagonal terms in the same row and column.[159]

We next consider the relationships among the slopes of marginal cost functions c_i, slopes of the spot market SFs β_i^s, the parameter ϕ_i, and the derivative $\partial \overline{\Sigma}_i^s \left(p^s ; \overline{q}_i^f , \overline{q}_j^f \right) \big/ \partial \overline{q}_i^f$ from eq. (5.13). Begin by considering the case of symmetric costs in which $c_1 = c_2$ in the definition (5.4) for ϕ_i. In this case, the symmetric forms of eqs. (5.6) and (5.11) imply that we must have $\beta_1^s = \beta_2^s$. From eq. (5.7), as a consequence, this symmetric scenario implies further that $\phi_1 = \phi_2$. We may therefore write that

$$c_1 = c_2 \quad \Rightarrow \quad \beta_1^s = \beta_2^s \quad \Rightarrow \quad \phi_1 = \phi_2 . \tag{5.17}$$

Moreover, using the equations of section 5.2, we may begin with any one of the equations in (5.17) to generate the other two equations given there. We thus may strengthen the implications in statement (5.17) to "if and only if" relationships as follows:

$$c_1 = c_2 \quad \Leftrightarrow \quad \beta_1^s = \beta_2^s \quad \Leftrightarrow \quad \phi_1 = \phi_2 . \tag{5.18}$$

Finally, we may generalize the statement (5.18) further to include asymmetric firms 1 and 2. Consider the two asymmetric cases $c_1 > c_2$ and $c_1 < c_2$ and the implications of each for

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[159] The inequalities (5.14) and (5.15) imply that such diagonal dominance holds for $n = 2$. We conjecture that this property holds more generally for $n > 2$.

the relative magnitudes of the β_i^s and the ϕ_i. Appealing to the signs of the partial derivatives from Table 5.1, to eq. (5.13), and to inequality (5.15) permits us to generalize (5.18) for the case of asymmetry in the following natural way:

$$c_1 \begin{pmatrix} > \\ = \\ < \end{pmatrix} c_2 \quad \Leftrightarrow \quad \beta_1^s \begin{pmatrix} < \\ = \\ > \end{pmatrix} \beta_2^s \quad \Leftrightarrow \quad \phi_1 \begin{pmatrix} < \\ = \\ > \end{pmatrix} \phi_2 \quad \Leftrightarrow \quad \frac{\partial \bar{\Sigma}_1^s}{\partial \bar{q}_1^f} \begin{pmatrix} < \\ = \\ > \end{pmatrix} \frac{\partial \bar{\Sigma}_2^s}{\partial \bar{q}_2^f}. \tag{5.19}$$

An implication of the statement (5.19) is that, loosely speaking, a high-cost firm is less able to affect the quantity that it bids in the spot market (at any given price) via its forward market position than is a low-cost firm. To put it another way, as a firm's cost increases, its quantity bid into the spot market, in equilibrium, becomes less sensitive to its forward market position.[160]

Figure 5.1 below depicts firm *i*'s spot market supply function $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ consistent with eqs. (5.9) and (5.12) for each of the two firms.[161]

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[160] Consistent with the statement (5.19), the terminology used here of "high-cost" and "low-cost" firms denotes, more precisely, the *slope* c_i of a firm's marginal cost function.

[161] The geometry of Figure 5.1 is consistent with Green's (1999a, 114) Figure 1 in which he considers spot market competition (also in affine SFs) in the presence of a forward contract market based on conjectural variations.

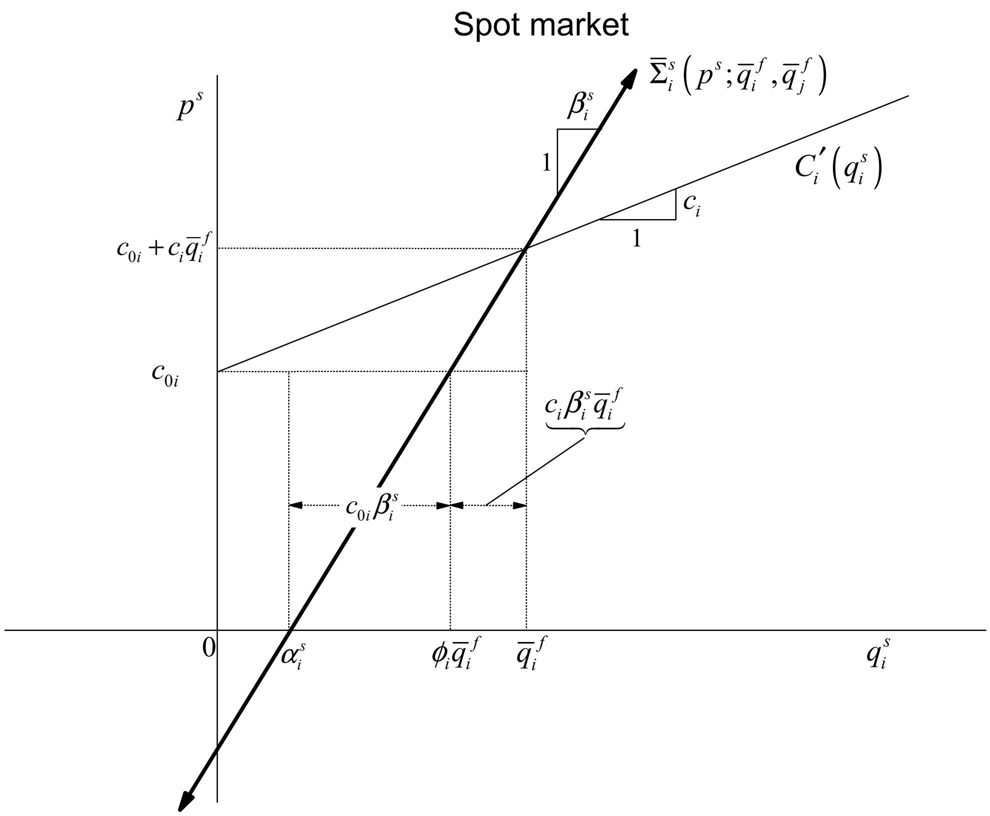

FIGURE 5.1: THE GEOMETRY OF THE SPOT MARKET SUPPLY FUNCTION $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$

Of particular interest in Figure 5.1 is the relationship of the SF $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ to the marginal cost function $C_i'\left(q_i^s\right)$ and firm *i*'s forward market quantity $\bar{q}_i^f$. As Green (1999a, 114) shows, the function $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ intersects $C_i'\left(q_i^s\right)$ at the point $\left(\bar{q}_i^f, C_i'\left(\bar{q}_i^f\right)\right)$. Consequently, increasing $\bar{q}_i^f$ translates the function $\bar{\Sigma}_i^s\left(p^s;\bar{q}_i^f,\bar{q}_j^f\right)$ horizontally to the right (recall eq. (5.13)), increasing firm *i*'s spot market bid quantity at every price p^s.[162] Another implication of Figure 5.1's geometry is that firm *i* bids its

[162] Allaz and Vila (1993) provide useful intuition for this effect of firms' forward market positions increasing their spot market quantities (also manifested in eq. (5.13)). Namely, in a model having a Cournot spot market, these authors find (p. 4) that "the decrease in price necessary to sell [an] additional

spot market quantity below its marginal cost at quantities below $\overline{q}_i^f$, and above its marginal cost at quantities above $\overline{q}_i^f$.

Figure 5.1 above is also useful to illustrate how the spot market geometry changes with shocks to the underlying parameters of interest.[163] In particular, consider the effects, in turn, of shocks to

- the marginal cost function intercepts c_{0i} and c_{0j},
- the marginal cost function slopes c_i and c_j, and
- the slope γ^s of the affine spot market demand function

on the functions depicted in Figure 5.1 for firm *i*. Assume, for simplicity, throughout this paragraph that firm *i*'s forward market quantity is fixed at $\overline{q}_i^f$.[164] Considering first an increase in the intercept c_{0i}, this shock induces an upward translation of both firm *i*'s marginal cost function $C_i'\left(q_i^s\right)$ and spot market SF $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$. In contrast, a shock to c_{0j} leaves the functions in Figure 5.1 unchanged. A shock to the slope c_i rotates the

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unit [on the spot market] does not affect the revenue from the forward sales." In other words, the marginal revenue function rotates counterclockwise about its price intercept, and the optimal spot market quantity increases. The same effect is present in this SF-based model.

[163] This discussion relies on the comparative statics effects of Table 5.1 on β_i^s and the definitions of firms' marginal cost functions $C_i'\left(q_i^s\right)$ (eq. (5.1)) and SFs $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ (eqs. (5.9) and (5.12)) above. See also Table E.1 of Appendix E.4 for corresponding numerical results from an affine example.

[164] A consequence of this assumption is that the point of intersection of firm *i*'s marginal cost function $C_i'\left(q_i^s\right)$ and spot market SF $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ remains fixed at the quantity $q=\overline{q}_i^f$, though the price $C_i'\left(\overline{q}_i^f\right)=c_{0i}+c_i\overline{q}_i^f$ at which this point of intersection occurs shifts, naturally, with shocks to c_{0i} or c_i.

function $C_i'\left(q_i^s\right)$ counterclockwise about its intercept c_{0i}, while both rotating the SF $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ counterclockwise and translating it upward. If instead we increase the slope c_j of firm *j*'s marginal cost function, this leaves the function $C_i'\left(q_i^s\right)$ unchanged, while rotating the SF $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ counterclockwise (but to a lesser degree than given a comparable shock to c_i, due to inequality (5.14)). Finally, consider the effect of a shock to γ^s, the magnitude of the spot market demand function's slope. A shock that increases γ^s makes this demand function (not shown in Figure 5.1) less steeply-sloped. This shock likewise makes $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ less steeply-sloped, rotating this SF clockwise, but leaves the function $C_i'\left(q_i^s\right)$ unchanged.

Green (1999a, 109) observed that "[a] general conjecture might be that as the spot market becomes more competitive, an uncompetitive contract market will have less impact on it" [footnote omitted]. As a final remark on the comparative statics results of Table 5.1, we obtain results from the multi-settlement SFE model that further support Green's conjecture above. Namely, consider again the effects of a change in γ^s, the magnitude of the spot market demand function's slope. Increasing γ^s leads both to (1) less steeply-sloped spot market SFs $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$, and (2) a decrease in ϕ_i, which we may interpret (from eq. (5.13)) as the sensitivity of (either firm's) SF $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ to (own) forward market quantity $\overline{q}_i^f$. That is, Table 5.1's results for marginal changes to γ^s imply that a change in the slope of spot market demand causing firms to behave *more* *competitively* in the spot market makes spot market actions (and hence the spot market outcome) *less sensitive* to forward market actions and outcomes (and vice-versa).

When firms compete in SFs in the spot market, they compete in an infinite-dimensional action space. In this case, strategic interaction in a duopoly cannot be completely characterized by using reaction functions in the plane, which assumes a one-dimensional strategy (or action) space for each firm. Even restricting firms' action spaces to affine spot market SFs as we do in this chapter, such action spaces are not one-dimensional, but two-dimensional. In this affine case, the firms' action spaces comprise, naturally, the slopes and intercepts of the affine SFs. As we have noted, we may solve eqs. (5.6) and (5.11) for the SF slopes $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$. That is, given exogenous values for c_i, c_j, and γ^s, the SF slopes β_i^s are independent of the intercepts α_i^s; in particular, the β_i^s are independent of forward market quantities $\overline{q}_i^f$. This property motivates the construction below of what we call *partial reaction functions* $R_i\left(\beta_j^s\right) \equiv \beta_i^s$ in the β_1^s - β_2^s plane. These partial reaction functions capture that portion of firms' responses to changes in the parameters c_i, c_j, and γ^s reflected in the slopes of the affine spot market SFs.[165] If we assume functional relationships between β_i^s and β_j^s, we may plot the partial reaction functions $R_1\left(\beta_2^s\right)$ and $R_2\left(\beta_1^s\right)$ using eqs. (5.6), (5.11), and (5.4) for ϕ_i ($i,\ j = 1, 2;\ i \neq j$), as done in Figure 5.2 below.

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[165] Similarly, Laussel (1992) interpreted the *slope* of an affine SF as the relevant strategic variable in a strategic international trade model.

Appendix D.3 demonstrates from these three equations that the partial reaction functions $R_i\left(\beta_j^s\right)$ have the form depicted in Figure 5.2. In particular, each function $R_i\left(\beta_j^s\right)$ is everywhere increasing and concave in its argument $\beta_j^s > 0$, with a positive β_i^s-axis intercept (in the limit). The slope $R_i'\left(\beta_j^s\right)$ takes on its maximum value at the β_i^s-axis intercept, decreasing as β_j^s increases and going to zero as $\beta_j^s \to \infty$. Consistent with these relationships, we find that for fixed c_i, β_i^s is bounded above by $1/c_i$. The unique intersection of the partial reaction functions $R_1\left(\beta_2^s\right)$ and $R_2\left(\beta_1^s\right)$ in the positive orthant corresponds, naturally, to firms' equilibrium choices of β_1^s and β_2^s.

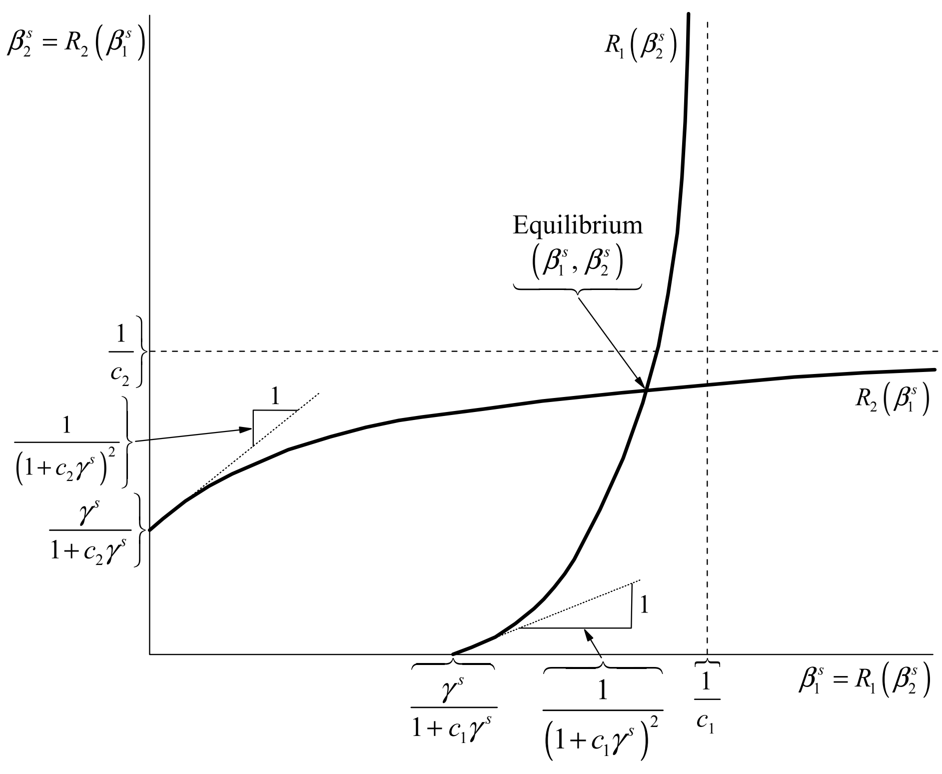

FIGURE 5.2: PARTIAL REACTION FUNCTIONS $R_i\left(\beta_j^s\right) \equiv \beta_i^s$ IN THE β_1^s-β_2^s PLANE: THE SPOT MARKET SUPPLY FUNCTION SLOPES β_1^s AND β_2^s ARE STRATEGIC COMPLEMENTS

Treating the slopes β_i^s as the strategic variable for each firm in the spot market, we may view the β_i^s as *strategic complements* in the sense of Bulow, Geanakoplos, and Klemperer (1985), since $R_i'\left(\beta_j^s\right) > 0$ for $i, j = 1, 2;\ i \neq j$. The "complementary" relationship between β_1^s and β_2^s implies, for example, that if firm 1 were to choose—for whatever reason—a steeper SF (a lower β_1^s), firm 2's best response would be to likewise submit a steeper SF (a lower β_2^s). Similarly, the best response to a flatter SF is likewise an SF with a flatter slope.

5.4 Implications for the optimal spot market price function

The spot market-clearing condition, given equilibrium forward quantities $\overline{q}_1^f$ and $\overline{q}_2^f$ and a realization of the spot market demand shock ε^s, is

$$\overline{\Sigma}_1^s\left(p^s;\overline{q}_1^f,\overline{q}_2^f\right)+\overline{\Sigma}_2^s\left(p^s;\overline{q}_2^f,\overline{q}_1^f\right)=D^s\left(p^s,\varepsilon^s\right). \tag{5.20}$$

The spot market-clearing price p^s satisfying eq. (5.20) is a function of both ε^s and $\overline{q}_i^f$, that is, $p^s \equiv p^{s*}\left(\varepsilon^s;\overline{q}_1^f,\overline{q}_2^f\right)$.[166] Using the Affine Spot Market Demand Function and Affine Spot Market SFs assumptions introduced at the outset of this chapter, eq. (5.20) becomes

$$\left[\left(\phi_1\overline{q}_1^f - c_{01}\beta_1^s\right)+\beta_1^s p^s\right]+\left[\left(\phi_2\overline{q}_2^f - c_{02}\beta_2^s\right)+\beta_2^s p^s\right]=-\gamma^s p^s+\varepsilon^s$$

which, solving for p^s, yields

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[166] Where it causes no ambiguity, we use the more convenient notation p^s in what follows.

$$p^s = \frac{\varepsilon^s - \phi_1 \bar{q}_1^f - \phi_2 \bar{q}_2^f + c_{01}\beta_1^s + c_{02}\beta_2^s}{\beta_1^s + \beta_2^s + \gamma^s}. \tag{5.21}$$

Given our assumptions, we have from eq. (5.21) that

$$\frac{\partial p^s}{\partial \bar{q}_i^f} = -\frac{\phi_i}{\beta_1^s + \beta_2^s + \gamma^s} < 0, \tag{5.22}$$

that is, an increase in either firm's forward market position decreases the equilibrium spot market price, ceteris paribus. For concreteness, consider an increase in $\bar{q}_1^f$, which from inequality (5.22) causes a decrease in the equilibrium price p^s. If firm 2's affine spot market SF remains unchanged, since this SF is assumed to be strictly increasing, the lower price causes firm 2 to reduce its spot market quantity offered. This is the same as Green's (1999a, 116) observation for affine SFs that one firm's quantity is decreasing in the other firm's forward market position, although the first firm's SF is unaffected.

Since we have from eq. (5.21) that p^s is affine in ε^s, we conclude that $p^{s*}\left(\varepsilon^s; \bar{q}_1^f, \bar{q}_2^f\right)$ is, in fact, partially invertible with respect to ε^s in the simplified affine example. In a Nash equilibrium, this implies that our earlier assumption (see section 4.1) of the partial invertibility of $p_i^{s*}\left(\varepsilon^s; \hat{q}_i^f, \hat{q}_j^f\right)$ with respect to ε^s is justified for the simplified affine example. More generally, due to the continuity of the underlying differential equations' solutions in the initial conditions,[167] this property of partial invertibility will hold also for spot market SFs sufficiently close to the affine SFs in eqs. (5.9) and (5.12).

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[167] See section 7.2.2 for a statement of the relevant continuity theorems for the forward market SFs. A similar argument holds here for the spot market SFs.

Replacing $\bar{q}_i^f$ with $\bar{S}_i^f\left(p^f\right)$, $i=1,2$, in eq. (5.21), we may also write this equation as a function of the forward market price p^f,

$$p^s = \frac{\varepsilon^s - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right) + c_{01}\beta_1^s + c_{02}\beta_2^s}{\beta_1^s + \beta_2^s + \gamma^s}. \tag{5.23}$$

It will be useful to simplify eq. (5.23) and the expressions that follow by defining some additional notation. Namely, let

$$\omega_a \equiv \frac{1}{\beta_1^s + \beta_2^s + \gamma^s} > 0 \tag{5.24}$$

and

$$\omega_b \equiv c_{01}\beta_1^s + c_{02}\beta_2^s \geq 0, \tag{5.25}$$

using subscript letters "$_a$" and "$_b$" to avoid confusion with firms 1 and 2. The signs of ω_a and ω_b above follow from the analysis of sections 5.2 and 5.3 and our parametric assumptions. Using the notation of eqs. (5.24) and (5.25), we may recast eq. (5.23) as

$$p^s = \omega_a\left[\varepsilon^s - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right) + \omega_b\right]. \tag{5.26}$$

Figure 5.3 below illustrates the clearing of the spot market and determination of the equilibrium price p^s, assuming affine marginal cost and spot market demand functions, and affine SFs (as depicted in Figure 5.1 for firm *i*).

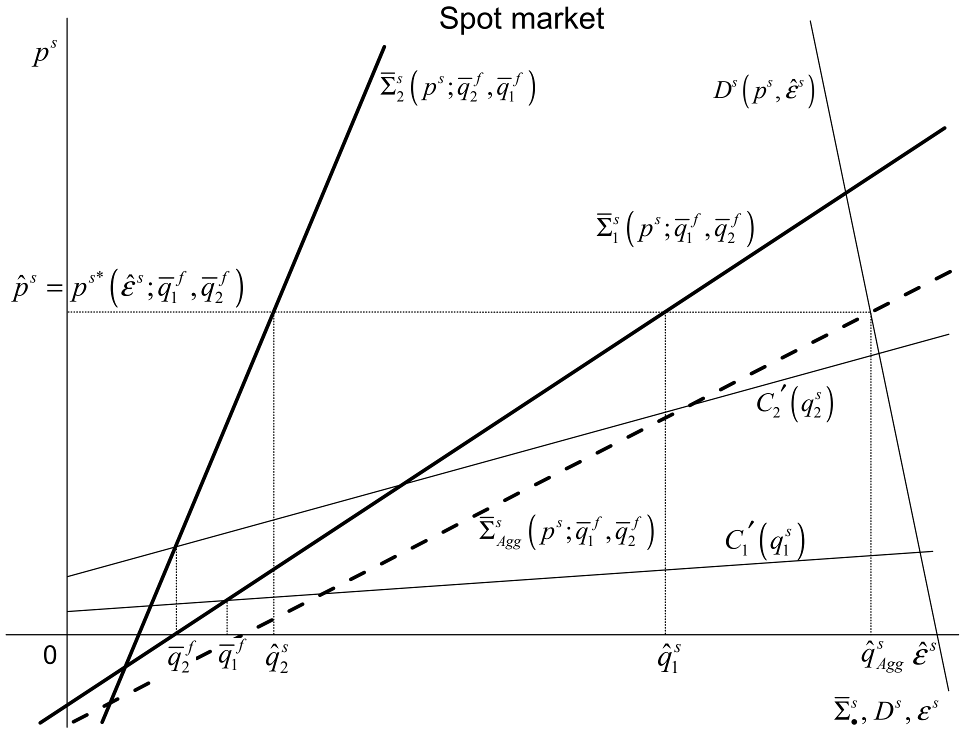

FIGURE 5.3: SPOT MARKET EQUILIBRIUM $\left(\hat{q}_{Agg}^s, \hat{p}^s\right)$ ASSUMING AFFINE FUNCTIONAL FORMS, AND GIVEN FORWARD MARKET QUANTITIES $\overline{q}_1^f$ AND $\overline{q}_2^f$ AND A SPOT MARKET DEMAND SHOCK $\varepsilon^s = \hat{\varepsilon}^s$

Given forward market quantities $\overline{q}_1^f$ and $\overline{q}_2^f$ and a spot market demand shock $\varepsilon^s = \hat{\varepsilon}^s$, Figure 5.3 illustrates how the firms' spot market SFs sum horizontally to yield the *aggregate spot market SF* $\overline{\Sigma}_{Agg}^s\left(p^s;\overline{q}_1^f,\overline{q}_2^f\right)$. The intersection of this function with spot market demand $D^s\left(p^s,\varepsilon^s\right)$, naturally, defines the equilibrium point $\left(\hat{q}_{Agg}^s, \hat{p}^s\right)$ for the spot market.

Returning to eq. (5.26) for p^s, we next compute the conditional expectation of this expression. Conditional on the forward market outcomes of the demand shock[168] ε_0^f and the corresponding market-clearing price p^f, this expectation is

$$\mathrm{E}\left(p^s \middle| p^f, \varepsilon_0^f\right) = \omega_a\left[\mathrm{E}\left(\varepsilon^s \middle| p^f, \varepsilon_0^f\right) - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right) + \omega_b\right]. \tag{5.27}$$

The quantities p^f and ε_0^f are related, naturally, in any forward market equilibrium. Chapter 4's optimization problem for firm 1 established the existence of an optimal forward market price function $p_1^{f*}\left(\varepsilon_0^f\right)$ for firm 1, and if $p_1^{f*}\left(\varepsilon_0^f\right)$ is invertible, an optimal SF $S_1^f\left(p^f\right)$.[169] Recall also that in Nash equilibrium, $p_1^{f*}\left(\varepsilon_0^f\right)$ and $p_2^{f*}\left(\varepsilon_0^f\right)$ must coincide in a *market-wide* optimal forward market price function, $p^{f*}\left(\varepsilon_0^f\right)$. Below, we establish sufficient conditions involving the forward market SFs for the invertibility and differentiability of $p^{f*}\left(\varepsilon_0^f\right)$. These properties will be useful later in simplifying eq. (5.27).

Consider equilibrium in the forward market. Given forward market equilibrium SFs $\bar{S}_i^f\left(p^f\right)$ and a demand function $D^f\left(p^f, \varepsilon_0^f\right)$ with arbitrary shock ε_0^f, the forward market clearing condition is (at a market-clearing price $p^f = p^{f*}\left(\varepsilon_0^f\right)$)[170]

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[168] Recall from eq. (3.10) that the additive forward market demand shock $\varepsilon_0^f = D^f\left(p_0^f, \varepsilon_0^f\right)$ is equal to the forward market demand function $D^f\left(p^f, \varepsilon_0^f\right)$ evaluated at the forward market reference price p_0^f.

[169] Symmetric results obtain, naturally, for firm 2.

[170] Under the assumptions of section 4.2, a unique market-clearing price will exist.

$$\bar{S}_1^f\left(p^f\right)+\bar{S}_2^f\left(p^f\right)=D^f\left(p^f,\varepsilon_0^f\right). \tag{5.28}$$

Substituting $p^{f*}\left(\varepsilon_0^f\right)$ for p^f in eq. (5.28) and recalling the additively separable form (eq. (3.8)) for the forward market demand function $D^f\left(p^f,\varepsilon_0^f\right)$, we may recast eq. (5.28) as

$$\bar{S}_1^f\left(p^{f*}\left(\varepsilon_0^f\right)\right)+\bar{S}_2^f\left(p^{f*}\left(\varepsilon_0^f\right)\right)=D_0^f\left(p^{f*}\left(\varepsilon_0^f\right)\right)+\varepsilon_0^f. \tag{5.29}$$

Since eq. (5.29) is an identity for each ε_0^f, and assuming that $\bar{S}_1^f(\bullet)$, $\bar{S}_2^f(\bullet)$, and $D_0^f(\bullet)$ are differentiable, we may totally differentiate eq. (5.29) with respect to ε_0^f to obtain

$$\bar{S}_1^{f\prime}\left(p^f\right)p^{f*\prime}\left(\varepsilon_0^f\right)d\varepsilon_0^f+\bar{S}_2^{f\prime}\left(p^f\right)p^{f*\prime}\left(\varepsilon_0^f\right)d\varepsilon_0^f=D_0^{f\prime}\left(p^f\right)p^{f*\prime}\left(\varepsilon_0^f\right)d\varepsilon_0^f+d\varepsilon_0^f.$$

Solving the above equation for $p^{f*\prime}\left(\varepsilon_0^f\right)$, we get

$$p^{f*\prime}\left(\varepsilon_0^f\right)=\frac{1}{\bar{S}_1^{f\prime}\left(p^f\right)+\bar{S}_2^{f\prime}\left(p^f\right)-D_0^{f\prime}\left(p^f\right)}. \tag{5.30}$$

We assume now that, in addition to being differentiable, the forward market SFs $\bar{S}_i^f(\bullet)$ are also strictly increasing[171] which, as chapter 6 will show, is sufficient for $\partial D^f\left(p^f,\varepsilon_0^f\right)/\partial p^f=D_0^{f\prime}\left(p^f\right)<0$. Then, we have from eq. (5.30) that $p^{f*}\left(\varepsilon_0^f\right)$ is differentiable and that

[171] Later, in the numerical examples of chapter 7, we will see that there exist forward market SFs satisfying these conditions, thus justifying this assumption.

$$p^{f*\prime}\left(\varepsilon_0^f\right) > 0 \tag{5.31}$$

for all $\varepsilon_0^f \in E^f$. The inequality (5.31) implies that the function $p^{f*}\left(\varepsilon_0^f\right)$ is invertible, so that we may define the function $e_p^f\left(p^f\right) = \varepsilon_0^f$ as the inverse of $p^{f*}\left(\varepsilon_0^f\right)$, that is,

$$\varepsilon_0^f = e_p^f\left(p^f\right) \equiv \left(p^{f*}\right)^{-1}\left(p^f\right). \tag{5.32}$$

Since the relationship $p^f = p^{f*}\left(\varepsilon_0^f\right)$ is invertible, p^f and ε_0^f are one-to-one. In eq. (5.27), therefore, we need condition on only one of the two quantities p^f and $\varepsilon_0^f = e_p^f\left(p^f\right)$. Conditioning on p^f alone, we may write eq. (5.27) as

$$\mathrm{E}\left(p^s \middle| p^f\right) = \omega_a\left[\mathrm{E}\left(\varepsilon^s \middle| e_p^f\left(p^f\right)\right) - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right) + \omega_b\right]. \tag{5.33}$$

Later, we use eq. (5.33) at the outset of chapter 7 to simplify the firms' forward market equilibrium optimality conditions (see also section 5.5 below).

In the multi-settlement SFE model, it is also of interest to determine how the expected spot market price $\mathrm{E}\left(p^s \middle| p^f\right)$ varies with marginal changes in forward market outcomes. To investigate this issue, we differentiate eq. (5.33) with respect to p^f to obtain

$$\frac{d\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f} = \omega_a\left[\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| e_p^f\left(p^f\right)\right)}{d\varepsilon_0^f} \cdot \frac{de_p^f\left(p^f\right)}{dp^f} - \phi_1 \bar{S}_1^{f\prime}\left(p^f\right) - \phi_2 \bar{S}_2^{f\prime}\left(p^f\right)\right]. \tag{5.34}$$

In the term $d\,\mathrm{E}\left(\varepsilon^s \middle| e_p^f\left(p^f\right)\right) \middle/ d\varepsilon_0^f$ on the right-hand side of eq. (5.34), we may condition on ε_0^f instead of $e_p^f\left(p^f\right)$ (recalling eq. (5.32)) for ease of notation. Making this change and recognizing also from eq. (5.32) that

$$\frac{de_p^f\left(p^f\right)}{dp^f} = \left[\frac{dp^{f*}\left(\varepsilon_0^f\right)}{d\varepsilon_0^f}\right]^{-1},$$

eq. (5.34) becomes

$$\frac{d\,\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f} = \omega_a \left\{ \left[\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} \middle/ \frac{dp^{f*}\left(\varepsilon_0^f\right)}{d\varepsilon_0^f} \right] - \phi_1 \bar{S}_1^{f\prime}\left(p^f\right) - \phi_2 \bar{S}_2^{f\prime}\left(p^f\right) \right\}. \quad (5.35)$$

Using eq. (5.30) to substitute for $p^{f*\prime}\left(\varepsilon_0^f\right) \equiv dp^{f*}\left(\varepsilon_0^f\right) \middle/ d\varepsilon_0^f$ in eq. (5.35) and collecting terms, we have

$$\begin{aligned} \frac{d\,\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f} = \omega_a \Bigg\{ & \left[\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} - \phi_1 \right] \bar{S}_1^{f\prime}\left(p^f\right) + \left[\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} - \phi_2 \right] \bar{S}_2^{f\prime}\left(p^f\right) \\ & - \frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} \cdot D_0^{f\prime}\left(p^f\right) \Bigg\}. \end{aligned} \quad (5.36)$$

To simplify eq. (5.36) further, the next chapter develops an expression for the derivative $d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right) \middle/ d\varepsilon_0^f$ assuming (1) a decomposition of ε^s into constituent stochastic parameters, and (2) a relationship between consumers' private information about the level of spot market demand, on the one hand, and forward market demand, on the other.

In interpreting the results of this subsection, it is important to note that we have not yet specified the forward market demand function. In particular, eq. (5.33) expresses the conditional expectation of p^s in terms of the forward market SFs, whose derivation in chapter 4 assumed the existence of a downward-sloping, twice-differentiable forward market demand function. We revisit this issue in chapter 6, in which we explain how such a forward market demand function could arise, and analyze this function's properties given the attributes of consumers.

5.5 Implications for the forward market optimality conditions

The Affine Spot Market Demand Function, Affine Marginal Production Cost Functions, and Affine Spot Market SFs assumptions permit us to simplify firm 1's equilibrium optimality condition for its forward SF, eqs. (4.41) and (4.42). From eq. (5.12), we have that

$$\frac{\partial \bar{\Sigma}_2^s\left(p^s;\bar{q}_2^f,\bar{q}_1^f\right)}{\partial \bar{q}_1^f}=0\,,$$

and

$$\frac{\partial \bar{\Sigma}_2^s\left(p^s;\bar{q}_2^f,\bar{q}_1^f\right)}{\partial \bar{q}_2^f}=\phi_2\,.$$

Using these expressions and after some simplification, eq. (4.41) becomes

$$\begin{aligned}&\left\{\phi_1\phi_2\left[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{01}+c_1\bar{S}_1^f\left(p^f\right)\right)\right]-\left[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\right]\right\}\bar{S}_2^{f\prime}\left(p^f\right)\\&\quad=\bar{S}_1^f\left(p^f\right)-D_0^{f\prime}\left(p^f\right)\left[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\right]\end{aligned}\tag{5.37}$$

for all market-clearing prices p^f.

For purposes of comparison with previous work, we make the temporary assumption that the expression in braces on the left-hand side of eq. (5.37) is nonzero, that is,

$$\phi_1\phi_2\left[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{01}+c_1\bar{S}_1^f\left(p^f\right)\right)\right]-\left[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\right]\neq 0. \tag{5.38}$$

This assumption permits us to rewrite eq. (5.37) as

$$\bar{S}_2^{f\prime}\left(p^f\right)=\frac{\bar{S}_1^f\left(p^f\right)-D_0^{f\prime}\left(p^f\right)\left[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\right]}{\phi_1\phi_2\left[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{01}+c_1\bar{S}_1^f\left(p^f\right)\right)\right]-\left[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\right]}. \tag{5.39}$$

Examining the right-hand side of eq. (5.39), we see that it depends on two price differences:

1. The difference between the expected spot price and the forward price, $\mathrm{E}\left(p^s\middle|p^f\right)-p^f$
2. The difference between the expected spot price and firm 1's marginal cost of producing its contract quantity in the forward market, $\mathrm{E}\left(p^s\middle|p^f\right)$ $-\left(c_{01}+c_1\bar{S}_1^f\left(p^f\right)\right)$

The structure of eq. (5.39) resembles that of KM's (single-market) optimality condition, namely,

$$S'(p)=\frac{S(p)}{p-C'\left(S(p)\right)}+D'(p). \tag{5.40}$$

The similarity between eqs. (5.39) and (5.40) is particularly apparent for the special case in which spot market demand $D^s\left(p^s,\varepsilon^s\right)$ is very elastic, so that its slope $-\gamma^s$ gets large in magnitude, that is, $-\gamma^s \to -\infty$. As $-\gamma^s$ decreases, we have from eq. (5.4) that $\phi_i \to 0$ ($i = 1, 2$). Setting $\phi_i = 0$ in eq. (5.39) as an approximation, we may then rewrite this equation as

$$\bar{S}_2^{f\prime}\left(p^f\right) = \frac{\bar{S}_1^f\left(p^f\right)}{p^f - \mathrm{E}\left(p^s \middle| p^f\right)} + D_0^{f\prime}\left(p^f\right). \tag{5.41}$$

Equation (5.41) is completely analogous to eq. (5.40), except that $\mathrm{E}\left(p^s \middle| p^f\right)$ appears in place of $C'\left(S\left(p\right)\right)$.[172] The structural similarity of these equations suggests that when spot market demand is perfectly elastic, the marginal opportunity cost of forward contract supply is simply the expected spot market price.

We may also derive a version of eq. (5.41) more directly if we solve firm 1's forward market problem (see chapter 4) with the simplifying assumptions that

1. the spot market price is random with expectation $\mathrm{E}\left(p^s\right)$, and
2. suppliers bid perfectly elastic supply functions that are independent of forward market outcomes.

We again assume (as justified in chapter 6) a downward-sloping forward market demand function (with shape component $D_0^f\left(p^f\right)$) given strictly increasing forward market SFs $\bar{S}_i^f\left(p^f\right)$. In this case, eq. (5.41) becomes

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[172] Earlier in chapter 4 (specifically, in discussing eqs. (4.41) and (4.42)), we took note of this analogy between expected spot price in the forward market problem and marginal cost in the (single market) spot market problem.

$$\bar{S}_2^{f\prime}\left(p^f\right) = \frac{\bar{S}_1^f\left(p^f\right)}{p^f - \mathrm{E}\left(p^s\right)} + D_0^{f\prime}\left(p^f\right), \tag{5.42}$$

which is identical to eq. (5.41), except that $\mathrm{E}\left(p^s\right)$ replaces $\mathrm{E}\left(p^s \middle| p^f\right)$, as a consequence of simplifying assumption 2 above.

5.6 Conclusion

This chapter assumed that cost functions, spot market demand functions, and spot market SFs have affine functional forms. These simplifications, naturally, have consequences for both the spot and forward markets which we explored in this chapter.

The next chapter, chapter 6, explains how the forward market demand function arises and investigates its properties. Then, chapter 7 will integrate the results of the present chapter, using eq. (5.33) for $\mathrm{E}\left(p^s \middle| p^f\right)$ to simplify further the forward market equilibrium optimality conditions (eq. (5.39) for firm 1, and analogously for firm 2), from which we derive the forward market SFE.

I can get no remedy against this consumption of the purse; borrowing only lingers and lingers it out, but the disease is incurable.

—Shakespeare, *Henry IV*, *Part 2*

Electricity seems destined to play a most important part in the arts and industries. The question of its economical application to some purposes is still unsettled, but experiment has already proved that it will . . . give more light than a horse.

—Ambrose Bierce, *The Devil's Dictionary*

6 The demand side

CONSUMERS purchase electricity for consumption in the spot market. In this chapter, we show under reasonable assumptions—notably, consumers' risk aversion—that consumers are also active in the *forward market*. Specifically, we derive here an *endogenous aggregate forward market demand function* for a representative consumer. Moreover, this chapter states sufficient conditions for this demand function—which we have denoted as $D^f\left(p^f, \varepsilon_0^f\right)$—to have the following properties.[173] First, $D^f\left(p^f, \varepsilon_0^f\right)$ *slopes downward* at all prices p^f, is *differentiable* with respect to both arguments, and its shape

[173] We already assumed some of these properties of $D^f\left(p^f, \varepsilon_0^f\right)$ in section 3.1.10. We may view the model presented in this chapter as justifying these assumptions.

is *deterministic* and *common knowledge*. Second, $D^f\left(p^f, \varepsilon_0^f\right)$ has an additive, exogenous, and stochastic component, subsumed in the shock ε_0^f, that shifts $D^f\left(p^f, \varepsilon_0^f\right)$ horizontally but does not change the function's shape (i.e., rotate or deform it).[174]

The outline of this chapter is as follows. We begin in section 6.1 by introducing some fundamental assumptions underlying the demand-side model. Section 6.2 motivates a nested optimization problem describing consumers' behavior in each market, and justifies a mean-variance approximation to consumers' utility maximization problem in the forward market. Next, section 6.3 gives sufficient conditions for a representative consumer to exist in the multi-settlement SFE model. Section 6.4 specifies attributes of the representative consumer that are consistent with an *affine* aggregate spot market demand function. Next, section 6.5 specifies a simple model for the spot market demand shock ε^s. Section 6.6 reframes the analysis in terms of a representative consumer. It then derives the representative consumer's forward and spot market activity as the solution to her underlying utility maximization problem.[175] In section 6.7, we show how demand shocks and prices are related across the two markets. Finally, section 6.8 characterizes the essential properties of the aggregate forward market demand function.

Several empirical studies have found that, as we might expect, estimated electricity forward market demand functions are downward sloping, and are more elastic

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[174] Many of these properties of the multi-settlement SFE model's forward market demand function are identical to those of KM's single-market (i.e., spot market) demand function. The shared features facilitate the application of KM's SFE framework to the forward market (in addition to the spot market) in the present work.

[175] As an arbitrary convention, we use feminine pronouns to denote consumers.

than typical estimates of spot market demand functions. For example, Earle (2000) studies the first twenty months of operation (i.e., from April 1998 to November 1999) of California's competitive market. Earle finds a downward-sloping residual demand function[176] with a median elasticity of approximately –0.1; in 27% of the hours in his data set, the magnitude of the residual demand elasticity exceeds one. Such values of demand elasticity are indeed markedly higher than short-run elasticities commonly measured in spot electricity markets. The present model's endogenous determination of $D^f\left(p^f, \varepsilon_0^f\right)$ naturally permits such elasticity, as well.

6.1 Modeling assumptions

This section outlines our assumptions regarding the attributes of consumers and motivates their optimization problems in the forward and spot markets.

6.1.1 Price-taking consumers

There are a total of J consumers active in the market, indexed by $j = 1, 2, \ldots, J$. We assume that J is *large* and *fixed*.[177] Furthermore, each consumer j is a *price taker* in both the forward and spot markets (consumers may be active in both markets).

6.1.2 Partial equilibrium analysis

Each consumer's expenditure on electricity is a small fraction of that consumer's total expenditures; this is also true with respect to each consumer's marginal expenditures. In

[176] Generating units in the California market which must run due to engineering constraints were bid into the PX with a (perfectly elastic and non-strategic) bid of zero dollars. Earle then subtracts such bids from total demand to obtain the residual demand function.

[177] We neglect the possibility of entry and exit of consumers, with the justification that these actions are costly.

addition, the electricity market, as such, is small relative to the entire economy. Hence, prices of other goods and services may be taken as approximately constant as the price of electricity varies. In this setting, *Marshallian partial equilibrium analysis* (Marshall 1920) implies for all consumers that (1) we may neglect wealth effects on electricity demand, and (2) we may treat expenditures on goods and services (other than electricity) as expenditures on a single composite commodity, termed the *numeraire* commodity (Mas-Collel, Whinston and Green 1995, 316) and denoted as m. Absent uncertainty, moreover, it is reasonable under partial equilibrium assumptions (Mas-Collel, Whinston and Green 1995, sec. 10.C) to take consumers' utility functions to be *quasilinear* with respect to this numeraire (implying no wealth effects for electricity demand, at least in the short run). We also assume a *utilitarian social welfare function*. Together, quasilinear utility functions and a utilitarian social welfare function imply that we may quantify changes in social welfare by measuring changes in aggregate surplus.[178]

6.1.3 A derived demand for electricity

Demand for energy (for example, electricity) is commonly considered a *derived demand*, as either an input to production[179] or a means to provide electricity-dependent services[180] (hereinafter *amenities*) to consumers. The consequence for the analysis of consumer behavior in the present model is that consumers' utility functions do not depend directly

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[178] Aggregate surplus (or *Marshallian* aggregate surplus) from consumption of a commodity is defined as the total utility generated by consumption of that commodity less its costs of production (Mas-Collel, Whinston and Green 1995, 326). Section 7.7 uses aggregate surplus to compute social welfare in the context of a specific numerical example.

[179] See, for example, Berndt and Wood's (1975) analysis.

[180] For example, light, heat, air conditioning, entertainment, etc.

on the amount of electricity consumed, but rather on the level of amenity enjoyed. A related element of the modeling framework adopted here is the assumption that each consumer notionally produces her amenity in a given market round using inputs of electricity and other (unmodeled) inputs, for example, capital/durable goods, labor/leisure time, assumed to be fixed.[181] The amount of amenity produced by the consumer is subject to stochastic shocks due, in turn, to environmental or technological factors.

We now introduce notation to characterize consumer *j*'s demand-side production process for her amenity. Define the following:

$q_j^s \in \mathbb{R}$ = consumer *j*'s quantity of electricity purchased in the spot market and subsequently used as an input to amenity production in a given market round;[182]

$m_j \in \mathbb{R}$ = consumer *j*'s consumption of the numeraire commodity *m*;[183]

$T_j \in \left[\underset{\smile}{T}_j, \widehat{T}_j\right] \subset \mathbb{R}$ = stochastic production shock with support $\left[\underset{\smile}{T}_j, \widehat{T}_j\right]$ characterizing randomness in consumer *j*'s production process due to environmental or technological factors;

$x_j \in \mathbb{R}$ = level of amenity[184] enjoyed by consumer *j*; and

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[181] The present approach is in the same spirit as Michael and Becker's (1973) reformulation of the theory of consumer behavior using a "household production function."

[182] An outcome in which $q_j^s < 0$ corresponds to consumer *j* being a net supplier of electricity in the given market round. This possibility could be interpreted as so-called "net metering," whereby consumers owning electricity generation capacity may sell electricity that they choose not to consume. While the present model permits this, in principle, the particular forward market equilibrium selection procedure employed in chapter 7's numerical examples preclude suppliers and consumers from switching sides in the spot market (but not in the forward market). See subsection 7.6.1 for details.

[183] We assume $m_j \in \mathbb{R}$ for convenience, to avoid boundary complications.

$f:\left(q_j^s, T_j\right) \rightarrow \mathbb{R}$ = consumer j's production function[185] relating the input q_j^s and the shock T_j to output.

Consumer j observes the realization of the stochastic shock T_j *before* selecting the optimal level of the input q_j^s to produce x_j. Assuming that no amenity is wasted (i.e., produced but not enjoyed), we may equate x_j and the amount produced as

$$x_j = f\left(q_j^s, T_j\right). \tag{6.1}$$

Next, assume that consumer j derives utility according to a utility function $W^s\left(m_j, x_j\right)$ from two sources in the spot market: (1) her consumption m_j of the numeraire commodity, and (2) her enjoyment of amenity x_j.[186] Let $W^s\left(m_j, x_j\right)$ be *quasilinear* with respect to j's consumption of the numeraire commodity, m_j, and let the contribution of

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[184] Since we take the units of the amenity x_j to be arbitrary for greatest generality, the origin of x_j is also arbitrary. Hence, x_j may be any real number.

[185] We specify the properties of the production function f below. While the function $f\left(q_j^s, T_j\right)$ is consumer-specific, we suppress its subscript "$_j$" to reduce notational clutter. The arguments q_j^s and T_j of $f\left(q_j^s, T_j\right)$ associate this function with consumer j. From note 184 and eq. (6.1), j's production $f\left(q_j^s, T_j\right)$ may be positive, negative, or zero.

[186] While the function $W^s\left(m_j, x_j\right)$ is consumer-specific, we suppress its subscript "$_j$" (as with f) to reduce notational clutter. The arguments m_j and x_j of $W^s\left(m_j, x_j\right)$ associate this function with consumer j.

x_j to *j*'s utility enter $W^s\left(m_j, x_j\right)$ as a function $\phi\left(x_j\right)$.[187] With the above assumptions, we may define $W^s\left(m_j, x_j\right)$ as

$$W^s\left(m_j, x_j\right) = m_j + \phi\left(x_j\right). \tag{6.2}$$

Turn now to the properties of the functions *f* and ϕ in expressions (6.1) and (6.2) above. First, let both *f* and ϕ be twice continuously differentiable in their arguments. Next, the conventional neoclassical theories of production and of demand—as well as the present modeling framework—suggest a number of a priori restrictions on the functional forms of both *f* and ϕ. Namely, we assume the following (letting subscripts denote partial differentiation) for the production function *f*:

- Production is (strictly) increasing in the input q_j^s (for q_j^s sufficiently small):[188]

$$f_{q_j^s}\left(q_j^s, T_j\right) > 0 \tag{6.3}$$

- Production is (strictly) decreasing in the production shock T_j:

$$f_{T_j}\left(q_j^s, T_j\right) < 0 \tag{6.4}$$

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[187] While the function $\phi\left(x_j\right)$ is consumer-specific, we suppress its subscript "$_j$" (as with *f* and W^s) to reduce notational clutter. The argument x_j of $\phi\left(x_j\right)$ associates this function with consumer *j*.

[188] We require this qualification on values of q_j^s to accommodate a *quadratic* functional form for the production function $f\left(q_j^s, T_j\right)$, which we specify in subsection 6.4.2.

- The marginal product of the input q_j^s is nonincreasing:

$$f_{q_j^s q_j^s}\left(q_j^s, T_j\right) \leq 0 \tag{6.5}$$

- The marginal product of the input q_j^s is (strictly) increasing in T_j:

$$f_{q_j^s T_j}\left(q_j^s, T_j\right) > 0 \tag{6.6}$$

Recall that by the argument of note 185, we have not restricted production to be nonnegative. For example, given the derivatives of *f* above, $x_j = f\left(0, T_j\right)$ might be negative for sufficiently large T_j, although this may not be an equilibrium outcome (see section 6.4.4). Now assume the following regarding the function ϕ:

- Utility is (strictly) increasing in x_j:

$$\phi'\left(x_j\right) > 0 \tag{6.7}$$

- Marginal utility is nonincreasing in x_j:

$$\phi''\left(x_j\right) \leq 0 \tag{6.8}$$

To provide some intuition for the application of the demand-side production model outlined above, consider the following specific example. Suppose that consumer *j* has a derived demand for electricity, q_j^s, to operate a household climate control system producing the amenity of "a comfortable indoor environment" or simply, "comfort," denoted as x_j. Consumption of a greater amount of electricity produces a higher level of comfort, but at a (weakly) decreasing rate, reflecting diminishing returns in q_j^s consistent with inequalities (6.3) and (6.5) above. Regarding the production shock T_j, one might interpret this shock, roughly speaking, as "ambient temperature"; it is useful, however, if we construe T_j more generally as any adverse shock in the ambient environment in real time. In the present example, increasing T_j *decreases* comfort, for any level of electricity consumption.[189] This is consistent with inequality (6.4) above. Finally, a larger value of the shock T_j increases the marginal productivity of the electricity input, meaning that an increment in electricity consumption produces more equivalent comfort at the margin, as inequality (6.6) indicates.

6.2 Consumers' optimization problems

This section develops a model of consumer *j*'s decisions in both the forward and spot markets. This model assumes that consumer *j*'s decisions maximize her (expected) utility from consumption of electricity and of the numeraire commodity. Previously, we noted that consumer *j* observes the realization of the stochastic shock T_j before making her spot market consumption decision. In modeling consumer *j*'s spot market problem, therefore, we may take T_j as given. In contrast, as consumer *j* faces her forward market problem, the real-time adverse environmental shock T_j is as yet unobserved. It is therefore appropriate to treat T_j as *stochastic* when modeling consumer *j*'s forward market decision making.

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[189] This flexible interpretation of T_j permits the demand-side production model to apply to most any residential (i.e., consumptive) or commercial (i.e., productive) use of electricity.

6.2.1 An expected utility maximization problem

First define the following additional notation:

$w_j^{ne} \in \mathbb{R}$	=	consumer *j*'s wealth endowment available for consumption, not including any proceeds from electricity market activity; in the partial equilibrium framework, it consists of an endowment of the numeraire commodity m (whereby the superscript "^{ne}" on w_j^{ne} indicates "non-electricity")
$q_j^f \in \mathbb{R}$	=	consumer *j*'s quantity of electricity purchased in the forward market[190]
p_m	=	the price of the numeraire commodity *m*

As in previous chapters, p^f and p^s are the electricity forward and spot market prices, respectively. We now define consumer *j*'s budget constraint for a given market round. In words, this budget constraint ensures that the sum of consumer *j*'s expenditures does not exceed the sum of her wealth available for consumption. In the multi-settlement model, consumer *j* incurs three distinct expenditures:

$p_m m_j$	=	expenditure on the numeraire commodity *m*,
$p^f q_j^f$	=	expenditure on electricity contracts in the forward market, and
$p^s q_j^s$	=	expenditure on electricity in the spot market,

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[190] Note that the domain for q_j^f is $\mathbb{R}$. Since forward contracts are settled financially and are not linked to electricity production, it is natural to permit consumers—and for that matter, suppliers as well—both to buy ($q_j^f > 0$, or a "long position") and sell ($q_j^f < 0$, or a "short position") in the forward market.

and has two sources of wealth available for consumption:

w_j^{ne} = non-electricity wealth endowment, and

$p^s q_j^f$ = settlement receipts in the spot market given forward contracts q_j^f

From the above discussion, we may write consumer *j*'s budget constraint algebraically as

$$p_m m_j + p^s q_j^s + p^f q_j^f \leq w_j^{ne} + p^s q_j^f ,$$

or collecting terms in q_j^f,

$$p_m m_j + p^s q_j^s \leq w_j^{ne} + \left(p^s - p^f \right) q_j^f . \tag{6.9}$$

In the budget constraint (6.9), we abstract from cashflows arising from shares that consumers may hold in the two supplier firms of the multi-settlement SFE model.[191] We may rationalize this assumption in two ways. The first potential justification is simply to assume that the firms are owned by agents other than the *J* consumers active on the demand side of the model. The second potential justification for this assumption is to permit such share ownership by consumers in the model, while supposing further that

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[191] Such shareholding is a standard element of models of competitive equilibrium (Mas-Collel, Whinston and Green 1995, 314).

consumers ignore the effects of their share ownership on forward market behavior[192] due to *bounded rationality*.[193]

Assume that consumer *j*'s objective in the spot market is to maximize her utility function $W^s\left(m_j, x_j\right)$ through optimal choices of m_j and q_j^s for consumption. For a given shock T_j, and using the maximand in eq. (6.2), the production constraint (6.1), and the budget constraint (6.9), we may write consumer *j*'s spot market optimization problem as

$$\begin{aligned} \max_{\substack{m_j \in \mathbb{R} \\ q_j^s \in \mathbb{R}}} \quad & m_j + \phi\left(x_j\right) \\ \text{s.t.} \quad & x_j = f\left(q_j^s, T_j\right) && \text{(production constraint)} \\ & p_m m_j + p^s q_j^s \le w_j^{ne} + \left(p^s - p^f\right) q_j^f && \text{(budget constraint).} \end{aligned} \tag{6.10}$$

Two simplifications to problem (6.10) are possible. First, we may substitute for x_j in this problem's objective function from the production constraint, since it is an equality. Second, it is evident that the budget constraint, as well, will hold with *equality* at any solution to this problem. Consequently, we may solve the budget constraint as an equality for m_j (taking $p_m = 1$ without loss of generality), and substitute for this variable

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[192] Only forward market behavior would be affected by the presence of cashflows from shareholding. In the forward market, such cashflow would be a random variable (a function of spot market uncertainty) that would covary with the spot market price p^s and hence affect the behavior of risk-averse consumers. In consumers' spot market problem, in contrast, such cashflows are treated as lump-sum receipts, fixed for a given p^s. As subsection 6.6.1 below elaborates, consumers optimize in the spot market conditional on p^s.

[193] The bounded rationality of consumers in this setting may be defended, in turn, by assuming that consumers' shareholding is intermediated (through, say, mutual funds). In such a case, the instantaneous exposure of consumers to the cashflows of the electricity suppliers may be relatively intransparent.

in problem (6.10)'s objective function. With these two simplifications, we may rewrite the spot market problem (6.10) as

$$\max_{q_j^s \in \mathbb{R}} \quad w_j^{ne} - p^s q_j^s + \left(p^s - p^f\right) q_j^f + \phi\left(f\left(q_j^s, T_j\right)\right). \tag{6.11}$$

Consumer *j* faces problem (6.11) in period 2 (recall Figure 3.1) after the forward market has cleared (revealing p^f and q_j^f), but before the spot market has cleared. Now consider consumer *j*'s *forward market* decision in period 1, given that she will face problem (6.11) in period 2. We need to augment problem (6.11) to provide a basis for her forward market decision making. To do so, we add three features to consumer *j*'s problem:

1. We introduce uncertainty in the parameter T_j (consistent with the discussion at the outset of this section).
2. We permit consumer *j* to assign a preference ranking[194] to the optimal outcomes of problem (6.11).[195]
3. We allow consumer *j* to maximize this preference ranking through her choice of forward quantity q_j^f (as a function of p^f, as we will see).

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[194] The function $W^s\left(m_j, x_j\right)$ introduced in eq. (6.2) accounts only for utility from consumption of electricity and the numeraire commodity. In contrast, the preference ranking sought here will take into account not only these utility terms, but will weigh their value along with changes in wealth due to electricity market activity, as well, via the terms $-p^s q_j^s$ and $\left(p^s - p^f\right) q_j^f$ in problem (6.11).

[195] Note that with the introduction of uncertainty in T_j, the outcomes of problem (6.11) are now themselves uncertain.

The optimal outcomes from problem (6.11) will be in monetary units ("wealth"),[196] so it is natural to assume that there exists a consistent functional representation of consumer *j*'s preference ranking of this problem's optimal outcomes. We denote such a function as $V(\bullet)$, defined over forward and (optimal) spot market outcomes from problem (6.11).[197] Applying $V(\bullet)$ to problem (6.11), we may write

$$V\left\{\max_{q_j^s \in \mathbb{R}}\left[w_j^{ne} - p^s q_j^s + \left(p^s - p^f\right) q_j^f + \phi\left(f\left(q_j^s, T_j\right)\right)\right]\right\}. \tag{6.12}$$

Consumption of numeraire produces utility directly, while according to the demand-side production model introduced in section 6.1.3, consumer *j* uses electricity as an input to produce x_j, whose enjoyment then contributes to her utility.

As an illustration, let $V(\bullet)$ be a *negative exponential function* of the form

$$V(z) = 1 - e^{-\lambda_j z},\ z \in \mathbb{R}, \tag{6.13}$$

with risk aversion parameter $\lambda_j > 0$. Note that $V(z)$ is strictly risk averse for all *z*, since $V''(z) < 0$. This form of utility function is also commonly referred to as the *constant absolute risk aversion*—or "CARA"—utility function, since the Arrow-Pratt absolute risk aversion coefficient, $r_A(z)$, for the utility function $V(z)$ of eq. (6.13) is constant:

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[196] Note that we may express the optimized value of $W^s(m_j, x_j)$ as an equivalent money metric indirect utility function.

[197] While the function $V(\bullet)$ is consumer-specific, we suppress its subscript "$_j$" (as with *f*, W^s, and ϕ) to reduce notational clutter. The arguments in $V(\bullet)$ (see problem (6.12)) associate this function with consumer *j*. We assume that $V(\bullet)$ is at least twice differentiable.

$$r_A(z) \equiv -\frac{V''(z)}{V'(z)} = \lambda_j. \tag{6.14}$$

In eq. (6.14), denote the parameter λ_j as the "CARA coefficient" for consumer *j*.[198] For ease of presentation, we continue below to refer to the function $V(\bullet)$ rather than use explicitly the negative exponential functional form of (6.13), although we will appeal in what follows to the properties of this functional form.

Assume that consumer *j* maximizes her expected utility of wealth, that is, she maximizes (with respect to q_j^f) the expectation $\mathrm{E}_j V(\bullet)$ in the forward market. Recasting problem (6.12) to reflect this objective, we have

$$\max_{q_j^f \in \mathbb{R}} \quad \mathrm{E}_j V\left\{\max_{q_j^s \in \mathbb{R}} \left[w_j^{ne} - p^s q_j^s + \left(p^s - p^f\right) q_j^f + \phi\left(f\left(q_j^s, T_j\right)\right)\right]\right\}. \tag{6.15}$$

We may simplify problem (6.15) further by noting the following:

1. Both consumer *j*'s non-electricity wealth endowment w_j^{ne} and (due to the price-taking assumption) the term $\left(p^s - p^f\right) q_j^f$ are independent of q_j^s, the decision variable for the inner maximization problem. Therefore, we may bring these two terms outside of the inner maximization problem.

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198 A useful intuitive interpretation of the CARA coefficient λ_j is as follows. Suppose that consumer *j* is offered a lottery paying τ_j with probability $\frac{1}{2}$ and $-\tau_j/2$ with probability $\frac{1}{2}$. If this consumer has a CARA utility function (e.g., eq. (6.13)) with CARA coefficient λ_j, it is straightforward to show that the value of τ_j for which *j* is indifferent between accepting and not accepting the given lottery is (approximately) the reciprocal of λ_j, that is, $\tau_j \approx 1/\lambda_j$. In this setting, we may interpret τ_j as consumer *j*'s *risk tolerance*. Pratt (1964, 126) offers another characterization of the coefficient of absolute risk aversion in terms of a probability premium for accepting a lottery.

2. Similar to the institutional structure on the supply side, both the forward and spot markets can accept bids from consumers in the form of a demand *function*, so that consumer *j*'s chosen quantity in market *m*, q_j^m, may, in fact, vary with price p^m ($m = f, s$). Since consumer *j* is a price taker, p^m is *exogenous* from her perspective. As a consequence, it is appropriate to condition market *m*'s objective function on an arbitrary p^m.

Making these changes in the forward market problem (6.15) yields[199]

$$\max_{q_j^f \in \mathbb{R}} \quad \mathrm{E}_j V\left\{\left[w_j^{ne} + \left(p^s - p^f\right)q_j^f + \max_{q_j^s \in \mathbb{R}}\left(\phi\left(f\left(q_j^s, T_j\right)\right) - p^s q_j^s\right)\right] \middle| p^f\right\}. \qquad (6.16)$$

We now allow for *asymmetric information* on the part of individual consumers. Assume that, before bidding in the forward market (i.e., during period 1), each consumer *j* observes a *private, random signal* $\eta_j \in \mathbb{R}_+$ that is informative concerning *j*'s subjective conditional probability distribution of p^s given p^f.[200] Hence, in problem (6.16), we condition expected utility $\mathrm{E}_j V(\bullet)$ on η_j, as well, to obtain

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[199] Problem (6.16) suggests a natural way to introduce *speculators* into the model, that is, demand-side agents who, rather than consuming electricity, simply speculate on the difference between the forward and spot market prices. Namely, if agent $\hat{j}$ were a speculator, we would constrain $q_{\hat{j}}^s = 0$ and $\phi\left(f\left(q_{\hat{j}}^s, T_{\hat{j}}\right)\right) = 0$, since by definition, the speculator $\hat{j}$ does not consume electricity (or produce the amenity *x*) and hence does not participate in the spot market. With these restrictions, problem (6.16) would become simply $\max_{q_{\hat{j}}^f} \mathrm{E}_{\hat{j}} V\left\{\left(p^s - p^f\right)q_{\hat{j}}^f \middle| p^f\right\}$, whereby speculator $\hat{j}$ chooses $q_{\hat{j}}^f$ to maximize his expected utility of profits as a function of p^f. By varying p^f, we would generate speculator $\hat{j}$'s forward market demand function.

[200] Since only consumer *j* observes the signal η_j, it is reasonable to suppose that market participants other than consumer *j* treat η_j as *stochastic*. We may think of η_j as representing any proprietary information available only to consumer *j*, such as competitive intelligence on other market

$$\max_{q_j^f \in \mathbb{R}} \quad \mathrm{E}_j V\left\{\left[w_j^{ne} + \left(p^s - p^f\right) q_j^f + \max_{q_j^s \in \mathbb{R}} \left(\phi\left(f\left(q_j^s, T_j\right)\right) - p^s q_j^s\right)\right] \middle| \left(\eta_j, p^f\right)\right\}. \quad (6.17)$$

Because consumer j is a price taker, the equilibrium spot market price p^s does not depend on q_j^s in eq. (6.17), but does depend on both η_j and p^f.

6.2.2 *Approximating the expected utility maximization problem with a mean-variance decision model*

In general, to compute the expectation in problem (6.17) exactly, we would need to resort to numerical integration,[201] since the distribution of the argument of $V(\bullet)$ is a non-trivial transformation of the distribution of T_j. Here we follow a more tractable (if approximate) approach to problem (6.17)—a *mean-variance decision model*[202]—which may yield a reasonable approximation to the exact solution of problem (6.17). There are several settings in which the use of a mean-variance decision model is exactly consistent with expected utility maximization, and others in which a mean-variance model can serve, at the least, as a good approximation of the expected utility maximization problem. This subsection examines these issues further, and justifies the use of a mean-variance approach to approximate the problem (6.17).

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participants, market research, or specialized weather forecasts that would help shape her spot price expectations for a particular market round.

[201] Alternatively, one could also apply Monte Carlo methods to obtain an arbitrarily close approximation to an exact solution.

[202] That is, a model in which an agent's decisions are based only on the mean and variance of the agent's payoff function and the form of the agent's utility function.

Denote consumer *j*'s payoff from a given market round (given p^f and η_j) as z_j (including, for convenience, the endowment w_j^{ne}). In problem (6.17), therefore, $z_j = z_j\left(q_j^f\right)$ is the expression within the braces, that is,[203]

$$z_j = z_j\left(q_j^f\right) \equiv \left[w_j^{ne} + \left(p^s - p^f\right) q_j^f + \max_{q_j^s \in \mathbb{R}} \left(\phi\left(f\left(q_j^s, T_j\right)\right) - p^s q_j^s \right) \right] \Big| \left(\eta_j, p^f\right). \quad (6.18)$$

Before the spot market clears, revealing p^s, z_j is itself a random variable whose distribution is a transformation of the (unspecified) distribution of T_j. Using the definition (6.18), we may write problem (6.17) concisely as

$$\max_{q_j^f \in \mathbb{R}} \quad \mathrm{E}_j V\left(z_j\left(q_j^f\right)\right). \quad (6.19)$$

We first note two cases in which a mean-variance decision model is exactly consistent with expected utility maximization. The first case having this property is one in which the underlying utility function has a quadratic functional form. Another example of such exact consistency is when the utility function is of the negative exponential form and, in addition, the payoffs are normally distributed (Freund 1956, 255). While we could assume (recall eq. (6.13)) that the utility function $V(\bullet)$ in problem (6.19) is indeed of the negative exponential form, the distribution of payoffs z_j is likely

[203] To reduce clutter in the following analysis, we suppress the dependence of z_j on q_j^f where it causes no confusion.

to be highly non-normal.[204] Thus, given the strong premises of Freund's result, it is not reasonable to appeal to it here.

Beyond the rather restrictive conditions for exact consistency between expected utility maximization and the mean-variance model, a growing strand of the finance literature has explored the conditions under which the mean-variance model serves as a reasonable approximation of expected utility maximization. In this work, the portfolio selection problem has naturally attracted much attention.[205] Grauer and Hakansson (1993, 859) surveyed this literature and concluded that "the consensus . . . is that portfolios chosen on the basis of mean and variance can closely approximate portfolios chosen by maximizing expected utility, especially when investors have similar risk aversion characteristics." More recent work (see, e.g., Amilon 2001) has confirmed the earlier findings,[206] lending support to the argument that the mean-variance model often leads to good approximations to the expected utility maximization result for empirical distributions.

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[204] Note that we have not specified the distribution of T_j itself, and moreover, do not need to do so for the present analysis. Since we assumed in section 6.1.3 that $T_j \in \left[\underset{\sim}{T}_j, \widehat{T}_j \right]$, however, whatever distribution one might choose for T_j on this bounded support would likely be highly non-normal. The related distribution of consumer *j*'s payoffs z_j would be a transformation of T_j's distribution.

[205] Indeed, researchers have examined this question since the middle of the twentieth century: Markowitz (1952) first applied the mean-variance model to the portfolio selection problem.

Note, however, that the payoff function (6.18) is clearly not that of a portfolio, which would be simply a weighted sum of (random) asset returns. Therefore, analytical and empirical results from the portfolio selection context are not directly applicable to the consumer's forward market problem (problem (6.19)) in the multi-settlement SFE model.

[206] Using a historical distribution of stock returns (shown *not* to be multivariate normally distributed), Amilon (2001) examined the portfolio selection problem. He found certainty equivalent losses of only a few percent for the mean-variance decision model compared to expected utility maximization for a wide variety of utility functions.

Returning to the multi-settlement SFE model (problem (6.19)), we develop below a simple mean-variance decision model of consumer *j*'s optimization problem in the forward market. Begin by writing the second-order Taylor series approximation to consumer *j*'s utility function $V\left(z_j\right)$ in the neighborhood of the expected value of z_j, $\bar{z}_j \equiv \mathrm{E}_j\left(z_j\right)$,

$$V\left(z_j\right) \approx V\left(\bar{z}_j\right) + \left(z_j - \bar{z}_j\right) V'\left(\bar{z}_j\right) + \frac{1}{2}\left(z_j - \bar{z}_j\right)^2 V''\left(\bar{z}_j\right). \tag{6.20}$$

The expected value of this approximation is

$$\mathrm{E}_j V\left(z_j\right) \approx V\left(\bar{z}_j\right) + \frac{1}{2} V''\left(\bar{z}_j\right) \mathrm{Var}_j\left(z_j\right). \tag{6.21}$$

The approximation (6.21) is a widely-used specification of a mean-variance model (Levy and Markowitz 1979). For our purposes, however, we further simplify this model via an additional approximation. Namely, we approximate the term $V\left(\bar{z}_j\right)$ in (6.21) with a first-order Taylor series approximation in the neighborhood of an arbitrary point z_j^0 in the support of z_j that is sufficiently close to—but distinct from—$\bar{z}_j$. Thus, we have

$$V\left(\bar{z}_j\right) \approx V\left(z_j^0\right) + \left(\bar{z}_j - z_j^0\right) V'\left(z_j^0\right). \tag{6.22}$$

Substituting (6.22) into (6.21) and rearranging yields

$$\mathrm{E}_j V\left(z_j\right) \approx V\left(z_j^0\right) - z_j^0 V'\left(z_j^0\right) + \bar{z}_j V'\left(z_j^0\right) + \frac{1}{2} V''\left(\bar{z}_j\right) \mathrm{Var}_j\left(z_j\right).$$

Dividing by $V'\left(z_j^0\right) > 0$ and writing the term $\bar{z}_j$ as $\mathrm{E}_j\left(z_j\right)$, we have

$$\frac{\mathrm{E}_j V\left(z_j\right)}{V'\left(z_j^0\right)} \approx \frac{V\left(z_j^0\right)}{V'\left(z_j^0\right)} - z_j^0 + \mathrm{E}_j\left(z_j\right) + \frac{1}{2}\frac{V''\left(\overline{z}_j\right)}{V'\left(z_j^0\right)} \cdot \mathrm{Var}_j\left(z_j\right). \tag{6.23}$$

Since (1) we chose z_j^0 to be sufficiently close to $\overline{z}_j$ by assumption above, and (2) $V\left(z_j\right)$ is a smooth function (earlier assumed to be of the negative exponential form), we may make the additional approximation that

$$V'\left(z_j^0\right) \approx V'\left(\overline{z}_j\right). \tag{6.24}$$

Substituting the approximation (6.24) for $V'\left(z_j^0\right)$ in only the last term on the right-hand side of (6.23) gives us

$$\frac{\mathrm{E}_j V\left(z_j\right)}{V'\left(z_j^0\right)} \approx \frac{V\left(z_j^0\right)}{V'\left(z_j^0\right)} - z_j^0 + \mathrm{E}_j\left(z_j\right) + \frac{1}{2}\frac{V''\left(\overline{z}_j\right)}{V'\left(\overline{z}_j\right)} \cdot \mathrm{Var}_j\left(z_j\right). \tag{6.25}$$

Recall that we defined consumer *j*'s CARA coefficient λ_j in eq. (6.14) given a negative exponential utility function $V\left(z_j\right) = 1 - e^{-\lambda_j z_j}$ (from eq. (6.13), letting $z = z_j$) as

$$\lambda_j \equiv -\frac{V''\left(z_j\right)}{V'\left(z_j\right)}. \tag{6.26}$$

Setting $z_j = \overline{z}_j$ in eq. (6.26) to substitute for $V''\left(\overline{z}_j\right) \big/ V'\left(\overline{z}_j\right)$ in (6.25), we may write (6.25) as

$$\frac{\mathrm{E}_j V\left(z_j\right)}{V'\left(z_j^0\right)} \approx \frac{V\left(z_j^0\right)}{V'\left(z_j^0\right)} - z_j^0 + \mathrm{E}_j\left(z_j\right) - \frac{\lambda_j}{2}\mathrm{Var}_j\left(z_j\right). \tag{6.27}$$

Multiplying both sides of (6.27) by $V'\left(z_j^0\right)$ yields

$$\mathrm{E}_j V(z_j) \approx \left[V(z_j^0) - z_j^0 V'(z_j^0)\right] + V'(z_j^0)\left[\mathrm{E}_j(z_j) - \frac{\lambda_j}{2}\mathrm{Var}_j(z_j)\right]. \tag{6.28}$$

Since the expressions $\left[V(z_j^0) - z_j^0 V'(z_j^0)\right]$ and $V'(z_j^0) > 0$ in (6.28) are constant, we may interpret $\mathrm{E}_j V(z_j)$ as an increasing function of the quantity $\left[\mathrm{E}_j(z_j) - \frac{\lambda_j}{2}\mathrm{Var}_j(z_j)\right]$. Accordingly, maximizing only the expression $\left[\mathrm{E}_j(z_j) \quad - \frac{\lambda_j}{2}\mathrm{Var}_j(z_j)\right]$ in (6.28) with respect to q_j^f will yield the same result as maximizing the entire right-hand side of (6.28). Therefore, from problem (6.19) (and recalling $z_j = z_j(q_j^f)$, again making the dependence of z_j on q_j^f explicit), we may write the optimal $q_j^f = q_j^{f*}$ from the maximization of expected utility $\mathrm{E}_j V(z_j(q_j^f))$ approximately as

$$q_j^{f*} \equiv \underset{q_j^f \in \mathbb{R}}{\arg\max} \quad \mathrm{E}_j V(z_j(q_j^f)) \approx \underset{q_j^f \in \mathbb{R}}{\arg\max} \quad \left[\mathrm{E}_j(z_j(q_j^f)) - \frac{\lambda_j}{2}\mathrm{Var}_j(z_j(q_j^f))\right]. \tag{6.29}$$

The problem (6.29) assumes that we will find approximately the same q_j^{f*} by (1) maximizing the expected utility of a payoff $\mathrm{E}_j V(z_j(q_j^f))$, as by (2) maximizing an additively separable function of only the payoff's *mean* $\mathrm{E}_j(z_j(q_j^f))$ and *variance* $\mathrm{Var}_j(z_j(q_j^f))$. The mean-variance model (6.29) has the following appealing properties:

1. It depends only on the first two moments of the distribution of z_j, and places *no restriction* on the nature of this distribution (e.g., z_j need not be—even approximately—normally distributed).

2. A generalized version of the result (6.29) would hold for other functional forms of $V_j(\cdot)$ (e.g., those not having the CARA property of eq. (6.14)) in which—recalling eqs. (6.25) and (6.26) with $z_j = \overline{z}_j$—the constant λ_j would be replaced by $-V''(\overline{z}_j) \big/ V'(\overline{z}_j)$.

At this point, we simply *assume* that the mean-variance decision model that underlies (6.29) yields acceptable approximations to consumer *j*'s expected utility maximization problem over the domain of interest. Naturally, when interpreting the results of the demand side analysis, one should bear in mind the various approximations—in particular, (6.20), (6.22), and (6.24) above—invoked in the course of this derivation.

In accordance with the above discussion, we recast consumer *j*'s expected utility maximization problem (6.17) as a mean-variance decision model over *j*'s payoffs, so that problem (6.17) becomes[207]

$$\begin{aligned} \max_{q_j^f \in \mathbb{R}} \quad & \Bigg(\mathrm{E}_j \left\{ \left[w_j^{ne} + \left(p^s - p^f\right) q_j^f + \max_{q_j^s \in \mathbb{R}} \left(\phi\left(f\left(q_j^s, T_j\right)\right) - p^s q_j^s \right) \right] \middle| \left(\eta_j, p^f\right) \right\} \\ & - \frac{\lambda_j}{2} \cdot \mathrm{Var}_j \left\{ \left[w_j^{ne} + \left(p^s - p^f\right) q_j^f + \max_{q_j^s \in \mathbb{R}} \left(\phi\left(f\left(q_j^s, T_j\right)\right) - p^s q_j^s \right) \right] \middle| \left(\eta_j, p^f\right) \right\} \Bigg). \end{aligned} \tag{6.30}$$

We later solve problem (6.30) for a *representative consumer* in section 6.6. The next section determines sufficient conditions for the existence of a representative consumer in the multi-settlement SFE model.

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[207] Problem (6.30) is essentially consistent with Bolle's (1993) characterization of a consumer's forward market problem, with the addition of a stochastic shock T_j in the spot market.

6.3 Existence of a representative consumer

To simplify the analysis,[208] we now demonstrate the existence of a notional *representative consumer* having forward and spot market demand functions that exhibit certain properties. It is useful to consider separately the forward and spot markets in this discussion, and also to distinguish between two senses of a representative consumer (following Mas-Collel, Whinston and Green 1995, 116)—a *positive representative consumer* (PRC) and a *normative representative consumer* (NRC). Below, we explain informally the meaning of these terms, and then explore sufficient conditions for existence of a representative consumer (in both the positive and normative senses above) in each of our two markets.

The former construct, the PRC, is intended to capture behavioral verisimilitude between all of the economy's consumers, on the one hand, and the PRC (if one exists), on the other. Informally,[209] we may say that there exists a PRC if we can specify a utility maximization problem for a fictitious individual—the putative PRC—whose solution would generate the economy's aggregate demand function. The latter construct, the NRC, presupposes the existence of a PRC (having an associated demand function), and in addition, requires that we be able to assign *welfare significance* to this demand function (Mas-Collel, Whinston and Green 1995, 116–117). Note that the existence of an NRC

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[208] The crucial simplification desired (in particular, for the analysis of section 6.6 below) is to abstract from the dependence of the shape of aggregate demand on the likely correlation among consumers' stochastic signals η_j. In the presence of such correlation, the functional form of $D^f\left(p^f, \varepsilon_0^f\right)$ (recall eq. (3.8)) would no longer be additively separable. Positing the existence of a representative consumer is one means of achieving this simplification.

[209] This informal definition is taken from Mas-Collel, Whinston, and Green (1995, 116), who provide (in their section 4.D) a comprehensive overview of representative consumer theory, including rigorous definitions of the PRC and the NRC. The informal definition above suffices for our purposes.

implies the existence of a PRC, so that it will be useful for our purposes to consider first the NRC, as we do in subsections 6.3.1 and 6.3.2 below.

6.3.1 A normative representative consumer in the forward market

In section 7.7, we compute a welfare measure for the multi-settlement SFE model while positing a risk-neutral social planner. Under this assumption, the spot market outcome contains all of the welfare-relevant information. Noting that the essence of the NRC is to define the attributes of a fictitious agent whose preferences can serve as a measure of aggregate welfare, we need not consider the question of the existence of the NRC in the forward market.

6.3.2 A normative representative consumer in the spot market

For simplicity, we rely on Mas-Collel, Whinston, and Green's (1995, 119) observation that the following two conditions are *sufficient* for the existence of an NRC:

1. Every consumer *j*'s indirect utility function $v_j\left(p, w_j\right)$ has the *Gorman form*, that is,

$$v_j\left(p, w_j\right) = a_j\left(p\right) + b\left(p\right) w_j, \tag{6.31}$$

   where *p* is the vector of prices in the economy, w_j is *j*'s total wealth, $v_j\left(p, w_j\right)$ is *j*'s indirect utility as a function of *p* and w_j, and $a_j\left(p\right)$ and $b\left(p\right)$ are functions of *p*.

2. The social welfare function is utilitarian.

That is, in the spot market of the multi-settlement SFE model, we may state the following: if the above conditions 1 and 2 hold, then spot market aggregate demand may always be interpreted as having been generated by an NRC (implying that the representative consumer's spot market demand function will be welfare-relevant).

In subsection 6.1.2, we already assumed that the social welfare function is utilitarian, thereby satisfying condition 2 above. As for condition 1, we argue that the model introduced in section 6.1 above implies that condition 1 holds, as well. Namely, taking consumers' spot market preferences to be *quasilinear* with respect to the numeraire commodity *m*—as we did in eq. (6.2)—implies that indirect utility $v_j\left(p, w_j\right)$ will be of the Gorman form (eq. (6.31)) with $b\left(p\right)=\dfrac{1}{p_m}$ (Mas-Collel, Whinston and Green 1995, 108 (n. 4)). Since conditions 1 and 2 above hold, we conclude that we may interpret *any* spot market aggregate demand function as having been generated by an NRC.

6.3.3 A positive representative consumer in the forward market

It may be shown that as the number of consumers *J* grows large, the influence of any individual consumer *j*'s private signal η_j wanes. To put it another way, as *J* grows, the conditional moments of shocks to spot market demand—conditional on an individual consumer's signal η_j—approach the corresponding unconditional moments, assumed to be common knowledge.[210] If *J* is sufficiently large so that the unconditional moments reasonably approximate the conditional moments of the demand shock, then we conjecture that, at least as an acceptable approximation, a PRC exists in the forward market.

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[210] For a specific model of the spot market demand shock ε^s, see section 6.5.

6.3.4 *A positive representative consumer in the spot market*

Since the existence of an NRC implies the existence of a PRC, we conclude from the argument of subsection 6.3.2 above that there exists a PRC in the spot market.

6.3.5 *Summary and conclusion*

Based on the discussion in the foregoing subsections, we assume now that

1. there exists an NRC and hence a PRC and in the spot market (see subsections 6.3.2 and 6.3.4) and
2. there exists a PRC in the forward market (see subsection 6.3.3).

For simplicity, we refer hereinafter to a "representative consumer" for the multi-settlement SFE model, and denote this consumer by "*R*" and likewise, subscript "$_R$."

The existence of the representative consumer *R* implies that we may solve *R*'s utility maximization problem to obtain her forward and spot market demand functions which are, identically, also *aggregate* demand functions for the *J* consumers.

6.4 Specification of functional forms for *f* and ϕ

This section seeks to identify functional specifications for

1. the representative consumer *R*'s production function, $x_R = f\left(q_R^s, T_R\right)$, and
2. *R*'s utility function, $\phi\left(x_R\right)$, for the amenity x_R

that yield a spot market demand function for R (identically, the *aggregate* spot market demand function) that is consistent with the affine spot market demand function in the simplified affine example first introduced in chapter 5,[211]

$$D^s\left(p^s, \varepsilon^s\right) = -\gamma^s p^s + \varepsilon^s . \tag{6.32}$$

In eq. (6.32), $\gamma^s > 0$ is a constant and ε^s is a stochastic parameter (with an as-yet-unspecified distribution).

Begin by defining the *composition* $\mathcal{C}$ of the functions $f\left(q_R^s, T_R\right)$ and $\phi\left(x_R\right)$ as

$$\mathcal{C}\left(q_R^s, T_R\right) \equiv \left(\phi \circ f\right)\left(q_R^s, T_R\right) \equiv \phi\left(f\left(q_R^s, T_R\right)\right). \tag{6.33}$$

The analysis of this section then proceeds as follows. Subsection 6.4.1 states *necessary and sufficient* conditions on $\mathcal{C}\left(q_R^s, T_R\right)$ from eq. (6.33) for the resulting spot market demand function for R—denoted as $D_R^s\left(p^s, T_R\right)$—to have the form of the affine spot market demand function (6.32). Next, in subsections 6.4.2 and 6.4.3, we specify individual functional forms for $f\left(q_R^s, T_R\right)$ and $\phi\left(x_R\right)$ that satisfy the a priori theoretical restrictions of subsection 6.1.3. Finally, subsection 6.4.4 then demonstrates that the assumed functional forms of f and ϕ are *sufficient* to ensure that $D_R^s\left(p^s, T_R\right)$ has the form of $D^s\left(p^s, \varepsilon^s\right)$ in eq. (6.32). In addition, we infer a simple relationship between the stochastic parameters ε^s and T_R.

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[211] See the Affine Spot Market Demand Function assumption, stated at the outset of chapter 5. This is also the form of the (single market) demand function assumed by Klemperer and Meyer (1989, 1260) in their "Linear Example."

6.4.1 *Necessary and sufficient conditions for the representative consumer* R *to have an affine spot market demand function*

This subsection states necessary and sufficient conditions for the representative consumer R to have an affine spot market demand function of the form of eq. (6.32). Begin with the representative consumer R's spot market problem, that is, the (identical) inner maximization problems of (6.30), conditioning on p^s (recalling subsection 6.2.1's argument) and letting $j = R$:

$$\max_{q_R^s \in \mathbb{R}} \left[\left(\phi\left(f\left(q_R^s, T_R \right) \right) - p^s q_R^s \right) \middle| p^s \right]. \tag{6.34}$$

Substituting for the functional composition $\phi\left(f\left(q_R^s, T_R \right) \right) = \mathcal{C}\left(q_R^s, T_R \right)$ from the definitions in the expression (6.33), problem (6.34) becomes

$$\max_{q_R^s \in \mathbb{R}} \left[\left(\mathcal{C}\left(q_R^s, T_R \right) - p^s q_R^s \right) \middle| p^s \right]. \tag{6.35}$$

The FOC corresponding to problem (6.35) is

$$\frac{\partial \mathcal{C}\left(q_R^s, T_R \right)}{\partial q_R^s} - p^s = 0 .$$

Defining $P_R^s\left(q_R^s, T_R \right)$ as R's *inverse spot market demand function* parameterized by the production shock T_R, the FOC becomes

$$P_R^s\left(q_R^s, T_R \right) \equiv \frac{\partial \mathcal{C}\left(q_R^s, T_R \right)}{\partial q_R^s} = p^s . \tag{6.36}$$

Next, denote the partial inverse of the function $P_R^s\left(q_R^s, T_R\right)$ in eq. (6.36) with respect to q_R^s as $\left(P_R^s\right)_{q_R^s}^{-1}\left(p^s, T_R\right)$.[212] The partial inverse of inverse demand with respect to quantity is simply R's (spot market) *demand function* (also parameterized by T_R) and denoted as $D_R^s\left(p^s, T_R\right)$:

$$D_R^s\left(p^s, T_R\right) \equiv \left(P_R^s\right)_{q_R^s}^{-1}\left(p^s, T_R\right). \tag{6.37}$$

We conclude that $f\left(q_R^s, T_R\right)$ and $\phi\left(x_R\right)$ are such that *R* has an affine spot market demand function of the form of eq. (6.32) *if and only if* $D_R^s\left(p^s, T_R\right)$ is of the form

$$D_R^s\left(p^s, T_R\right) = -\gamma_R^s p^s + g_R\left(T_R\right) \tag{6.38}$$

(given the definitions in (6.33), (6.36), and (6.37)), where $\gamma_R^s > 0$ is constant and $g_R(\bullet)$ is some differentiable function of T_R. Note that for any function $D_R^s\left(p^s, T_R\right)$ having the separable affine form of eq. (6.38), the partial inverse in eq. (6.37) indeed exists.

6.4.2 The representative consumer R*'s production function,* $f\left(q_R^s, T_R\right)$*, for the amenity* x_R

We now specify a functional form for $f\left(q_R^s, T_R\right)$. Together with a specification for $\phi\left(x_R\right)$ in the following subsection, these example specifications will be sufficient to

[212] Note that the notation "$\left(P_R^s\right)_{q_R^s}^{-1}$" in this expression denotes a *partial inverse* of P_R^s with respect to q_R^s, *not* partial differentiation. Following eq. (6.38), we check whether this partial inverse in fact exists.

ensure that the resultant spot market demand function for R is consistent with $D^s\left(p^s, \varepsilon^s\right)$ in eq. (6.32).

Let the representative consumer R's production function, $f\left(q_R^s, T_R\right)$, have the form

$$f\left(q_R^s, T_R\right) \equiv a_0 + a_1\left(q_R^s - T_R\right) - \frac{a_2}{2}\cdot\left(q_R^s - T_R\right)^2, \tag{6.39}$$

with coefficients $a_0, a_1, a_2 > 0$. Given the functional form in eq. (6.39) for $f\left(q_R^s, T_R\right)$, the a priori restrictions (6.3) and (6.4) are satisfied for (taking $q_R^s = q_R^{s*}$, R's *optimal* spot market quantity)[213]

$$q_R^{s*} - T_R < \frac{a_1}{a_2}, \tag{6.40}$$

while the a priori restrictions (6.5) and (6.6) always obtain.

6.4.3 *The representative consumer* R*'s utility function,* $\phi\left(x_R\right)$*, for the amenity* x_R

Let the representative consumer R's utility function for electricity consumption, $\phi\left(x_R\right)$, be linear in x_R, that is,

$$\phi\left(x_R\right) = b x_R, \quad b > 0. \tag{6.41}$$

[213] We revisit the condition (6.40) in subsection 6.4.4 below.

The linear functional form in eq. (6.41) for $\phi(x_R)$ is sufficient for the a priori restrictions (6.7) and (6.8) to hold.

The assumption that $\phi(x_R)$ is linear in x_R is a limiting case, used here for simplicity without loss of generality. As we may infer from the development of the necessary and sufficient condition in subsection 6.4.1, there is a tradeoff in the degree of concavity in the functions $\phi(x_R)$ and $f(q_R^s, T_R)$ (concavity with respect to q_R^s, in the case of $f(q_R^s, T_R)$) satisfying these conditions. Hence, we may make $\phi(x_R)$ concave while preserving the desired properties of the composition $(\phi \circ f)(q_R^s, T_R)$ by simultaneously decreasing the degree of concavity of $f(q_R^s, T_R)$. For example, given the functions $f(q_R^s, T_R)$ and $\phi(x_R)$ from eqs. (6.39) and (6.41), suppose that $\alpha \geq 1$ parameterizes a family of pairs of functions $f_\alpha(q_R^s, T_R) = \left[f(q_R^s, T_R)\right]^\alpha$ and $\phi_\alpha(x_R) = \phi\left((x_R)^{1/\alpha}\right)$. While the example in the text assumes $\alpha = 1$, a pair of such functions for any $\alpha > 1$ would also yield an affine spot market demand function for *R* of the form of eq. (6.38).

6.4.4 Conditions for consistency of $D^s(p^s, \varepsilon^s)$ *and* $D_R^s(p^s, T_R)$

As the analysis in subsection 6.4.1 demonstrates, the form of $D_R^s(p^s, T_R)$ depends on the specifications of $f(q_R^s, T_R)$ and $\phi(x_R)$. Substituting in the spot market problem (6.34) for the functions $f(q_R^s, T_R)$ and $\phi(x_R)$ from eqs. (6.39) and (6.41), respectively, yields

$$\max_{q_R^s \in \mathbb{R}} \left[\left(b\left(a_0 + a_1\left(q_R^s - T_R\right) - \frac{a_2}{2}\cdot\left(q_R^s - T_R\right)^2 \right) - p^s q_R^s \right) \middle| p^s \right]. \tag{6.42}$$

The FOC (for an interior solution) corresponding to problem (6.42) is

$$b\left(a_1 - a_2\left(q_R^s - T_R\right)\right) - p^s = 0\,. \tag{6.43}$$

Solving eq. (6.43) for the optimal[214] $q_R^s = q_R^{s*}$ as a function of p^s and T_R yields *R*'s spot market demand function $D_R^s\left(p^s, T_R\right)$,

$$q_R^s = q_R^{s*} \equiv D_R^s\left(p^s, T_R\right) = -\left(\frac{1}{a_2 b}\right)p^s + T_R + \frac{a_1}{a_2}\,. \tag{6.44}$$

By construction, $D_R^s\left(p^s, T_R\right)$ in the expression (6.44) has the separable affine form of eq. (6.38) where

$$\gamma_R^s = \frac{1}{a_2 b} > 0 \tag{6.45}$$

and

$$g_R\left(T_R\right) = T_R + \frac{a_1}{a_2}\,. \tag{6.46}$$

Because $D_R^s\left(p^s, T_R\right)$ in the expression (6.44) is a schedule of prices and *R*'s corresponding *optimal* quantities (given T_R), this function is useful in determining when

[214] We may also see from eq. (6.43) that given the assumed parameter restrictions, the second-order sufficient condition for a profit maximum will also hold.

the inequality (6.40) in subsection 6.4.2 above indeed holds. Rearranging the expressions in (6.44), we have that

$$q_R^{s*} - T_R = -\left(\frac{1}{a_2 b}\right) p^s + \frac{a_1}{a_2}. \tag{6.47}$$

Since $-\left(1/a_2 b\right) < 0$, it follows from eq. (6.47) that

$$p^s \begin{pmatrix} > \\ = \\ < \end{pmatrix} 0 \quad \Leftrightarrow \quad q_R^{s*} - T_R \begin{pmatrix} < \\ = \\ > \end{pmatrix} \frac{a_1}{a_2}. \tag{6.48}$$

When $p^s \le 0$, the expression (6.48) implies that the inequality (6.40) is violated. In this event, the a priori functional form restrictions (6.3) and (6.4) *do not* hold. While we do not rule out the event $p^s \le 0$ in the multi-settlement SFE model, we may choose parameter values to render nonpositive prices a relatively uncommon occurrence. Accordingly, we say that under "normal" circumstances, we have that $p^s > 0$, and therefore by the above argument, all of the a priori functional form restrictions (6.3)–(6.6) are normally satisfied.

By definition, *R* is the only consumer in the representative consumer model. Consequently, *R*'s spot market demand function $D_R^s\left(p^s, T_R\right)$ in eq. (6.44) is identically also the *aggregate* spot market demand function, $D^s\left(p^s, \varepsilon^s\right)$, in eq. (6.32), although these functions are parameterized differently by T_R and ε^s, respectively. Thus we have

$$D^s\left(p^s, \varepsilon^s\right) = D_R^s\left(p^s, T_R\right) \tag{6.49}$$

for every price p^s and production shock T_R. From eqs. (6.32), (6.38), and (6.44)–(6.46), eq. (6.49) implies that we must have the following two parametric restrictions for $D^s\left(p^s, \varepsilon^s\right)$ and $D_R^s\left(p^s, T_R\right)$ to be mutually consistent:

$$\gamma^s = \gamma_R^s = \frac{1}{a_2 b} > 0 \tag{6.50}$$

and

$$\varepsilon^s = g_R\left(T_R\right) = T_R + \frac{a_1}{a_2}. \tag{6.51}$$

Given a distribution for T_R and the parameters of the production function, eq. (6.51) indicates that the distribution of ε^s is a simple translation of the distribution of T_R. In particular, we may relate the support of ε^s to that of T_R as follows. Recalling that $\underset{\smile}{\varepsilon}^s$ and $\overset{\frown}{\varepsilon}^s$ are the lower and upper limits of the support of ε^s, $E^s \equiv \left[\underset{\smile}{\varepsilon}^s, \overset{\frown}{\varepsilon}^s\right]$, respectively, these limits are given by

$$\underset{\smile}{\varepsilon}^s = \underset{\smile}{T}_R + \frac{a_1}{a_2} \tag{6.52}$$

and

$$\overset{\frown}{\varepsilon}^s = \overset{\frown}{T}_R + \frac{a_1}{a_2}. \tag{6.53}$$

Finally, to simplify notation in the remainder of this chapter, we exploit eqs. (6.32) and (6.49) to rewrite eq. (6.44) for the optimal q_R^{s*} (conditional on p^s and ε^s) as simply the spot market aggregate demand function $D^s\left(p^s, \varepsilon^s\right)$,

$$q_R^{s*} \equiv D^s\left(p^s, \varepsilon^s\right) = -\gamma^s p^s + \varepsilon^s . \tag{6.54}$$

Equation (6.54) is the form of aggregate spot market demand that we posited in chapter 5's simplified affine example. In particular, $D^s\left(p^s, \varepsilon^s\right)$ is affine and downward-sloping.

6.5 A simple stochastic model for the spot market demand shock ε^s

We now specify a simple model for the spot market demand shock ε^s in terms of *R*'s stochastic signal η_R. Ultimately, this model will permit us to relate demand shocks and prices across the two markets. Begin by introducing a random variable ν_R that is revealed to *R* at $t = 2$ (see Figure 3.1), when the spot market clears with (public) revelation of the demand shock ε^s. Let ν_R be defined such that a simple additive relationship exists between the spot market demand shock ε^s on the one hand, and η_R and ν_R on the other. Namely, we have that

$$\varepsilon^s = \eta_R + \nu_R . \tag{6.55}$$

An intuitive interpretation of eq. (6.55) is that ν_R is a noise parameter whose presence makes *R*'s signal η_R an *imperfect signal* for ε^s.

Now consider the probability distributions of η_R and ν_R. Let η_R and ν_R be jointly distributed with a stationary distribution function $F_{\eta_R,\nu_R}\left(\eta_R, \nu_R\right)$, which we assume to be *common knowledge*. Further, let η_R and ν_R be *independent*, so that, denoting the marginal distributions of η_R and ν_R as $F_{\eta_R}\left(\eta_R\right)$ and $F_{\nu_R}\left(\nu_R\right)$, respectively, we have that $F_{\eta_R,\nu_R}\left(\eta_R, \nu_R\right) = F_{\eta_R}\left(\eta_R\right) \cdot F_{\nu_R}\left(\nu_R\right)$. In section 6.2.1, we took the stochastic support of η_j to be $\mathbb{R}_+$, so that with $j = R$, we have that $\eta_R \in \mathbb{R}_+$, in principle.[215] We now also let the stochastic support of ν_R be $\mathbb{R}_+$, in principle (see note 215). From eq. (6.55) and from independence, we then have that the support E^s of ε^s is, in principle[216]

$$E^s = [0, \infty). \tag{6.56}$$

Denote the means of η_R and ν_R as $\bar{\eta}_R \equiv \mathrm{E}(\eta_R)$ and $\bar{\nu}_R \equiv \mathrm{E}(\nu_R)$, and denote their variances as $\sigma^2_{\eta_R} \equiv \mathrm{Var}(\eta_R)$ and $\sigma^2_{\nu_R} \equiv \mathrm{Var}(\nu_R)$, respectively. Also, define the higher moment $\sigma_{\nu_R^2,\nu_R} \equiv \mathrm{Cov}(\nu_R^2, \nu_R)$.[217] In light of the independence assumption for η_R and ν_R, we may also interpret ν_R as that component of ε^s that is unexplained by (or orthogonal to) the signal η_R.

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[215] In practice, given eq. (6.55), a finite upper limit on the supports of both η_R and ν_R would delimit the extent of the corresponding spot market SFs.

[216] Equation (6.51) and the support of ε^s in eq. (6.56) imply that the support of T_R is, in principle, $T_R \in [\underset{\sim}{T}_R, \hat{T}_R] = [-a_1/a_2, \infty)$.

[217] To aid intuition concerning the higher moment $\sigma_{\nu_R^2,\nu_R} \equiv \mathrm{Cov}(\nu_R^2, \nu_R)$, we may show that $\sigma_{\nu_R^2,\nu_R} = (\sigma^2_{\nu_R})^{3/2}[\alpha_3 + (2/V_{\nu_R})]$, where $\alpha_3 \equiv m_3/(m_2)^{3/2}$ is the *coefficient of skewness* of ν_R, m_k is the k^{th} moment about the mean of ν_R (so that $m_2 = \sigma^2_{\nu_R} \equiv \mathrm{Var}(\nu_R)$, as defined above), and $V_{\nu_R} \equiv \sigma_{\nu_R}/\bar{\nu}_R$ is the *coefficient of variation* of ν_R. Recalling that positively-skewed distributions correspond to $\alpha_3 > 0$, and negatively-skewed distributions to $\alpha_3 < 0$, we may conclude the following concerning $\mathrm{sgn}(\sigma_{\nu_R^2,\nu_R})$:

1. If $\bar{\nu}_R \geq 0$ and the distribution of ν_R is positively skewed, then $\sigma_{\nu_R^2,\nu_R} > 0$.
2. If $\bar{\nu}_R \leq 0$ and the distribution of ν_R is negatively skewed, then $\sigma_{\nu_R^2,\nu_R} < 0$.
3. In all other cases, we may conclude only that $\mathrm{sgn}(\sigma_{\nu_R^2,\nu_R}) = \mathrm{sgn}[\alpha_3 + (2/V_{\nu_R})]$.

We next derive expressions for R's subjective conditional moments of ε^s, conditional on an arbitrary realization η_R of R's signal. First, denote R's subjective conditional expectation as $\mathrm{E}_R\left(\varepsilon^s \middle| \eta_R\right)$ where, from eq. (6.55) and using the notation introduced above, we have that

$$\mathrm{E}_R\left(\varepsilon^s \middle| \eta_R\right) = \eta_R + \mathrm{E}_R\left(\nu_R \middle| \eta_R\right) = \eta_R + \overline{\nu}_R . \tag{6.57}$$

The second equality in eq. (6.57) exploits both the independence of η_R and ν_R and the common knowledge distribution of ν_R. Similarly, denote R's subjective conditional variance as $\mathrm{Var}_R\left(\varepsilon^s \middle| \eta_R\right)$, which is

$$\mathrm{Var}_R\left(\varepsilon^s \middle| \eta_R\right) = \mathrm{Var}_R\left(\left(\eta_R + \nu_R\right) \middle| \eta_R\right) = \mathrm{Var}_R\left(\nu_R \middle| \eta_R\right) = \mathrm{Var}_R\left(\nu_R\right) = \sigma^2_{\nu_R} . \tag{6.58}$$

We use the results of eqs. (6.57) and (6.58) in subsection 6.6.2 below to simplify the expression for R's contribution to aggregate forward market demand as we solve the representative consumer R's maximization problem in the multi-settlement market setting.

6.6 The representative consumer *R*'s optimization problem

The sequential structure of the multi-settlement market problem implies that, as on the supply side, backward induction is the appropriate solution algorithm. Accordingly, subsection 6.6.1 considers the spot market in the first stage of the backward induction algorithm. Next, the second stage of the algorithm, discussed in subsection 6.6.2, addresses the forward market.

6.6.1 *Spot market*

The first stage of the backward induction algorithm is to solve the representative consumer *R*'s spot market problem, that is, the (identical) inner maximization problems of (6.30). We do so for a fixed T_R, which fixes ε^s (by eq. (6.51)), and for an arbitrary spot market price p^s. Accordingly, we condition on p^s, and let $j = R$ to obtain the spot market problem (see problem (6.34))

$$\max_{q_R^s \in \mathbb{R}} \left[\left(\phi\left(f\left(q_R^s, T_R \right) \right) - p^s q_R^s \right) \middle| p^s \right]. \tag{6.59}$$

In preparation for the forward market analysis in the next subsection, we may write problem (6.59) as follows (using eqs. (6.39) (6.41), and (6.44) for $f\left(q_R^s, T_R\right)$, $\phi\left(x_R\right)$, and q_R^{s*}, respectively):

$$\begin{aligned}
\max_{q_R^s \in \mathbb{R}} \left[\left(\phi\left(f\left(q_R^s, T_R \right) \right) - p^s q_R^s \right) \middle| p^s \right] &= \phi\left(f\left(q_R^{s*}, T_R \right) \right) - p^s q_R^{s*} \\
&= b\left(a_0 + a_1\left(q_R^{s*} - T_R \right) - \frac{a_2}{2} \cdot \left(q_R^{s*} - T_R \right)^2 \right) - p^s q_R^{s*} \\
&= b\left(a_0 + \frac{a_1^2}{2a_2} \right) - \frac{a_1 p^s}{a_2} - p^s T_R + \frac{\left(p^s\right)^2}{2a_2 b}.
\end{aligned}$$

Solving eq. (6.51) for $T_R = \varepsilon^s - \left(a_1/a_2\right)$, we may substitute this expression into the third equation above for T_R and simplify to obtain

$$\max_{q_R^s \in \mathbb{R}} \left[\left(\phi\left(f\left(q_R^s, T_R \right) \right) - p^s q_R^s \right) \middle| p^s \right] = b\left(a_0 + \frac{a_1^2}{2a_2} \right) - p^s \varepsilon^s + \frac{\left(p^s\right)^2}{2a_2 b}. \tag{6.60}$$

For notational convenience, define a constant *k* as

$$k \equiv b\left(a_0 + \frac{a_1^2}{2a_2}\right). \tag{6.61}$$

We may write eq. (6.60) more compactly by substituting from eq. (6.50) for $1/a_2 b$ and from eq. (6.61) for $b\left(a_0 + a_1^2/2a_2\right)$ to obtain

$$\max_{q_R^s \in \mathbb{R}} \left[\left(\phi\left(f\left(q_R^s, T_R\right)\right) - p^s q_R^s\right) \middle| p^s\right] = k - p^s \varepsilon^s + \frac{\gamma^s\left(p^s\right)^2}{2}. \tag{6.62}$$

The result in eq. (6.62) will be useful in the forward market analysis, to which we now turn.

6.6.2 *Forward market*

In the second stage of the backward induction algorithm, we analyze *R*'s forward market problem which, letting $j = R$, is the outer maximization problem of (6.30). Substituting from eq. (6.62) for *R*'s spot market surplus $\phi\left(f\left(q_R^{s*}, T_R\right)\right) - p^s q_R^{s*}$ (at an optimum) in problem (6.30), we have

$$\begin{aligned} \max_{q_R^f \in \mathbb{R}} \Bigg(& \mathrm{E}_R \left\{ \left(w_R^{ne} + \left(p^s - p^f\right) q_R^f + k - p^s \varepsilon^s + \frac{\gamma^s\left(p^s\right)^2}{2} \right) \middle| \left(\eta_R, p^f\right) \right\} \\ & - \frac{\lambda_R}{2} \cdot \mathrm{Var}_R \left\{ \left(w_R^{ne} + \left(p^s - p^f\right) q_R^f + k - p^s \varepsilon^s + \frac{\gamma^s\left(p^s\right)^2}{2} \right) \middle| \left(\eta_R, p^f\right) \right\} \Bigg). \end{aligned} \tag{6.63}$$

Distributing the expectation and variance operators in the problem (6.63), this expression becomes

$$
\begin{aligned}
\max_{q_R^f \in \mathbb{R}} \Bigg(& w_R^{ne} + k + q_R^f \left[\mathrm{E}_R \left(p^s \middle| \left(\eta_R, p^f \right) \right) - p^f \right] + \mathrm{E}_R \left[\left(-p^s \varepsilon^s + \frac{\gamma^s \left(p^s \right)^2}{2} \right) \middle| \left(\eta_R, p^f \right) \right] \\
& - \frac{\lambda_R}{2} \cdot \Big\{ \left(q_R^f \right)^2 \mathrm{Var}_R \left[p^s \middle| \left(\eta_R, p^f \right) \right] + \mathrm{Var}_R \left[p^s \varepsilon^s \middle| \left(\eta_R, p^f \right) \right] \\
& \qquad + \frac{\left(\gamma^s \right)^2}{4} \cdot \mathrm{Var}_R \left[\left(p^s \right)^2 \middle| \left(\eta_R, p^f \right) \right] - 2 q_R^f \, \mathrm{Cov}_R \left[\left(p^s, p^s \varepsilon^s \right) \middle| \left(\eta_R, p^f \right) \right] \\
& \qquad + q_R^f \gamma^s \, \mathrm{Cov}_R \left[\left(p^s, \left(p^s \right)^2 \right) \middle| \left(\eta_R, p^f \right) \right] - \gamma^s \, \mathrm{Cov}_R \left[\left(p^s \varepsilon^s, \left(p^s \right)^2 \right) \middle| \left(\eta_R, p^f \right) \right] \Big\} \Bigg).
\end{aligned}
$$

The FOC with respect to q_R^f for this maximization problem is (with some further simplification)

$$
\begin{aligned}
& \left[\mathrm{E}_R \left(p^s \middle| \left(\eta_R, p^f \right) \right) - p^f \right] \\
& \quad - \lambda_R \cdot \Big\{ q_R^f \, \mathrm{Var}_R \left(p^s \middle| \left(\eta_R, p^f \right) \right) - \mathrm{Cov}_R \left[\left(p^s, p^s \varepsilon^s \right) \middle| \left(\eta_R, p^f \right) \right] \\
& \qquad + \frac{\gamma^s}{2} \cdot \mathrm{Cov}_R \left[\left(p^s, \left(p^s \right)^2 \right) \middle| \left(\eta_R, p^f \right) \right] \Big\} = 0.
\end{aligned}
$$

Solving this condition for the optimal q_R^f as a function of p^f and η_R yields

$$
\begin{aligned}
q_R^f &= q_R^{f*} \left(p^f, \eta_R \right) \\
&= \frac{1}{\lambda_R \, \mathrm{Var}_R \left(p^s \middle| \left(\eta_R, p^f \right) \right)} \cdot \Big\{ \mathrm{E}_R \left(p^s \middle| \left(\eta_R, p^f \right) \right) - p^f \\
& \qquad + \lambda_R \, \mathrm{Cov}_R \left[\left(p^s, p^s \varepsilon^s \right) \middle| \left(\eta_R, p^f \right) \right] \\
& \qquad - \frac{\gamma^s \lambda_R}{2} \cdot \mathrm{Cov}_R \left[\left(p^s, \left(p^s \right)^2 \right) \middle| \left(\eta_R, p^f \right) \right] \Big\}.
\end{aligned}
\tag{6.64}
$$

Simplifying eq. (6.64) further, we examine, in turn, the two covariance terms and the expectation and variance terms on the right-hand side of this equation. To evaluate the two covariance terms, we first need to make explicit the dependence of p^s on ε^s.

Consistent with the statement of the suppliers' forward market problem in eqs. (4.16)–(4.18), the representative consumer R solves her forward market problem assuming equilibrium in the spot market. Therefore, it is appropriate at this point to take p^s in eq. (6.64) to be a *market-clearing* spot market price, given a spot market shock and the forward market outcome. Equation (5.26)—rewritten below as eq. (6.65)—gives an expression for the market-clearing price p^s as a function of ε^s and p^f:[218]

$$p^s \equiv p^{s*}\left(\varepsilon^s; \bar{S}_1^f\left(p^f\right), \bar{S}_2^f\left(p^f\right)\right) = \omega_a\left[\varepsilon^s - \phi_1\bar{S}_1^f\left(p^f\right) - \phi_2\bar{S}_2^f\left(p^f\right) + \omega_b\right]. \quad (6.65)$$

Now substitute from eq. (6.65) for p^s in $\mathrm{Cov}_R\left[\left(p^s, p^s\varepsilon^s\right)\middle|\left(\eta_R, p^f\right)\right]$, the first covariance term on the right-hand side of eq. (6.64):

$$\begin{aligned}
&\mathrm{Cov}_R\left[\left(p^s, p^s\varepsilon^s\right)\middle|\left(\eta_R, p^f\right)\right] \\
&\quad = \mathrm{Cov}_R\Big[\Big(\omega_a\left[\varepsilon^s - \phi_1\bar{S}_1^f\left(p^f\right) - \phi_2\bar{S}_2^f\left(p^f\right) + \omega_b\right], \\
&\qquad\qquad \omega_a\left[\varepsilon^s - \phi_1\bar{S}_1^f\left(p^f\right) - \phi_2\bar{S}_2^f\left(p^f\right) + \omega_b\right]\varepsilon^s\Big)\Big|\left(\eta_R, p^f\right)\Big] \\
&\quad = \mathrm{Cov}_R\left[\left(\omega_a\varepsilon^s, \omega_a\left(\varepsilon^s\right)^2\right)\middle|\eta_R\right] \\
&\qquad + \mathrm{Cov}_R\left[\left(\omega_a\varepsilon^s, \omega_a\left[\omega_b - \phi_1\bar{S}_1^f\left(p^f\right) - \phi_2\bar{S}_2^f\left(p^f\right)\right]\varepsilon^s\right)\middle|\eta_R\right],
\end{aligned}$$

which simplifies to

$$\begin{aligned}
\mathrm{Cov}_R\left[\left(p^s, p^s\varepsilon^s\right)\middle|\left(\eta_R, p^f\right)\right] &= \omega_a^2\,\mathrm{Cov}_R\left[\left(\varepsilon^s, \left(\varepsilon^s\right)^2\right)\middle|\eta_R\right] \\
&\quad + \omega_a^2\left[\omega_b - \phi_1\bar{S}_1^f\left(p^f\right) - \phi_2\bar{S}_2^f\left(p^f\right)\right]\mathrm{Var}_R\left(\varepsilon^s\middle|\eta_R\right).
\end{aligned} \quad (6.66)$$

---

[218] Recall from eqs. (5.24) and (5.25) that, in eq. (6.65), $\omega_a \equiv \left(\beta_1^s + \beta_2^s + \gamma^s\right)^{-1}$ and $\omega_b \equiv c_{01}\beta_1^s + c_{02}\beta_2^s$. That is, ω_a and ω_b are functions only of exogenous spot market parameters.

Next, substitute from eq. (6.65) for p^s in $\mathrm{Cov}_R\left[\left(p^s,\left(p^s\right)^2\right)\middle|\left(\eta_R, p^f\right)\right]$, the second covariance term on the right-hand side of eq. (6.64), and partially expand the square $\left(p^s\right)^2$:

$$
\begin{aligned}
&\mathrm{Cov}_R\left[\left(p^s,\left(p^s\right)^2\right)\middle|\left(\eta_R, p^f\right)\right] \\
&\quad = \mathrm{Cov}_R\Big[\Big(\omega_a\left[\varepsilon^s-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)+\omega_b\right], \\
&\qquad\qquad \omega_a^2\left(\varepsilon^s\right)^2+2\omega_a^2\left[\omega_b-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)\right]\varepsilon^s \\
&\qquad\qquad +\omega_a^2\left[\omega_b-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)\right]^2\Big)\Big|\left(\eta_R, p^f\right)\Big] \\
&\quad = \mathrm{Cov}_R\left[\left(\omega_a\varepsilon^s, \omega_a^2\left(\varepsilon^s\right)^2\right)\middle|\eta_R\right] \\
&\qquad + \mathrm{Cov}_R\left[\left(\omega_a\varepsilon^s, 2\omega_a^2\left[\omega_b-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)\right]\varepsilon^s\right)\middle|\eta_R\right],
\end{aligned}
$$

which simplifies to

$$
\begin{aligned}
&\mathrm{Cov}_R\left[\left(p^s,\left(p^s\right)^2\right)\middle|\left(\eta_R, p^f\right)\right] \\
&\quad = \omega_a^3\,\mathrm{Cov}_R\left[\left(\varepsilon^s,\left(\varepsilon^s\right)^2\right)\middle|\eta_R\right] \\
&\qquad + 2\omega_a^3\left[\omega_b-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)\right]\mathrm{Var}_R\left(\varepsilon^s\middle|\eta_R\right).
\end{aligned}
\tag{6.67}
$$

Now consider the expectation and variance terms on the right-hand side of eq. (6.64). We may evaluate these terms by taking the subjective conditional expectation and variance of p^s in eq. (6.65), again conditional on η_R and p^f. Doing so yields

$$
\mathrm{E}_R\left(p^s\middle|\left(\eta_R, p^f\right)\right)=\omega_a\left[\mathrm{E}_R\left(\varepsilon^s\middle|\eta_R\right)-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)+\omega_b\right] \tag{6.68}
$$

and

$$\mathrm{Var}_R\left(p^s \middle| \left(\eta_R, p^f\right)\right) = \omega_a^2 \,\mathrm{Var}_R\left(\varepsilon^s \middle| \left(\eta_R, p^f\right)\right) = \omega_a^2 \,\mathrm{Var}_R\left(\varepsilon^s \middle| \eta_R\right). \tag{6.69}$$

Using the results of eqs. (6.57) and (6.58) to simplify eqs. (6.68) and (6.69), these latter equations become

$$\mathrm{E}_R\left(p^s \middle| \left(\eta_R, p^f\right)\right) = \omega_a\left[\eta_R + \bar{v}_R - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right) + \omega_b\right] \tag{6.70}$$

and

$$\mathrm{Var}_R\left(p^s \middle| \left(\eta_R, p^f\right)\right) = \sigma_{v_R}^2 \omega_a^2. \tag{6.71}$$

Collecting the above results, we substitute from eqs. (6.66), (6.67), (6.70), and (6.71) into eq. (6.64) to obtain

$$\begin{aligned}
&q_R^{f*}\left(p^f, \eta_R\right) \\
&\quad = \frac{1}{\lambda_R \sigma_{v_R}^2 \omega_a^2} \cdot \Big\{\omega_a\left[\eta_R + \bar{v}_R - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right) + \omega_b\right] - p^f \\
&\qquad + \lambda_R \omega_a^2 \,\mathrm{Cov}_R\left[\left(\varepsilon^s, \left(\varepsilon^s\right)^2\right) \middle| \eta_R\right] \\
&\qquad + \lambda_R \omega_a^2 \left[\omega_b - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right)\right] \mathrm{Var}_R\left(\varepsilon^s \middle| \eta_R\right) \\
&\qquad - \frac{\lambda_R \gamma^s \omega_a^3}{2} \cdot \mathrm{Cov}_R\left[\left(\varepsilon^s, \left(\varepsilon^s\right)^2\right) \middle| \eta_R\right] \\
&\qquad - \lambda_R \gamma^s \omega_a^3 \left[\omega_b - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right)\right] \mathrm{Var}_R\left(\varepsilon^s \middle| \eta_R\right)\Big\}.
\end{aligned}$$

Collecting like terms, this equation becomes

$$\begin{aligned}
&q_R^{f*}\left(p^f, \eta_R\right) \\
&\quad = \frac{1}{\lambda_R \sigma_{v_R}^2 \omega_a^2} \cdot \Big\{\omega_a\left[\eta_R + \bar{v}_R - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right) + \omega_b\right] - p^f \\
&\qquad + \frac{\lambda_R \omega_a^2}{2} \cdot \left(2 - \gamma^s \omega_a\right) \mathrm{Cov}_R\left[\left(\varepsilon^s, \left(\varepsilon^s\right)^2\right) \middle| \eta_R\right] \\
&\qquad + \lambda_R \omega_a^2 \left(1 - \gamma^s \omega_a\right)\left[\omega_b - \phi_1 \bar{S}_1^f\left(p^f\right) - \phi_2 \bar{S}_2^f\left(p^f\right)\right] \mathrm{Var}_R\left(\varepsilon^s \middle| \eta_R\right)\Big\}.
\end{aligned} \tag{6.72}$$

From the definition of ε^s in eq. (6.55), we may evaluate the covariance term on the right-hand side of eq. (6.72) as

$$\begin{aligned}\mathrm{Cov}_R\left[\left(\left(\varepsilon^s\right)^2,\varepsilon^s\right)\middle|\eta_R\right] &= \mathrm{Cov}_R\left[\left(\left(\eta_R+\nu_R\right)^2,\eta_R+\nu_R\right)\middle|\eta_R\right]\\ &= \mathrm{Cov}_R\left[\left(\eta_R^2+2\eta_R\nu_R+\nu_R^2,\eta_R+\nu_R\right)\middle|\eta_R\right]\\ &= 2\eta_R\,\mathrm{Var}_R\left(\nu_R\middle|\eta_R\right)+\mathrm{Cov}_R\left[\left(\nu_R^2,\nu_R\right)\middle|\eta_R\right].\end{aligned}$$

Using the notation for the distributional moments introduced in section 6.5, we may write this term as

$$\mathrm{Cov}_R\left[\left(\left(\varepsilon^s\right)^2,\varepsilon^s\right)\middle|\eta_R\right]=2\eta_R\sigma_{\nu_R}^2+\sigma_{\nu_R^2,\nu_R}\,. \tag{6.73}$$

Finally, substituting from eqs. (6.73) and (6.58) for the covariance and variance terms, respectively, in eq. (6.72) and factoring ω_a out of the braces, we have

$$\begin{aligned}q_R^{f*}\left(p^f,\eta_R\right)=\frac{1}{\lambda_R\sigma_{\nu_R}^2\omega_a}\cdot\Bigg\{&\eta_R+\bar{\nu}_R-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)+\omega_b-\frac{p^f}{\omega_a}\\ &+\frac{\lambda_R\omega_a}{2}\left(2-\gamma^s\omega_a\right)\left(2\eta_R\sigma_{\nu_R}^2+\sigma_{\nu_R^2,\nu_R}\right)\\ &+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\left[\omega_b-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)\right]\Bigg\}.\end{aligned}$$

Collecting terms in p^f and η_R, we may write $q_R^{f*}\left(p^f,\eta_R\right)$ as

$$\begin{aligned}&q_R^{f*}\left(p^f,\eta_R\right)\\ &\quad=\frac{1}{\lambda_R\sigma_{\nu_R}^2\omega_a}\cdot\Bigg\{-\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\left[\phi_1\bar{S}_1^f\left(p^f\right)+\phi_2\bar{S}_2^f\left(p^f\right)\right]\\ &\qquad-\frac{p^f}{\omega_a}+\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\eta_R+\bar{\nu}_R+\frac{\lambda_R\sigma_{\nu_R^2,\nu_R}\omega_a}{2}\cdot\left(2-\gamma^s\omega_a\right)\\ &\qquad+\omega_b\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\Bigg\}.\end{aligned} \tag{6.74}$$

Since the forward market demand function of the representative consumer R is identically also the aggregate forward market demand function, denoted here as $q^{f*}\left(p^f,\eta_R\right)$, we have

$$q^{f*}\left(p^f,\eta_R\right)=q_R^{f*}\left(p^f,\eta_R\right). \tag{6.75}$$

Recall that in chapter 3 (eq. (3.8)), we expressed forward market demand as

$$D^f\left(p^f,\varepsilon_0^f\right)=D_0^f\left(p^f\right)+\varepsilon_0^f. \tag{6.76}$$

Equations (6.75) and (6.76) (using eq. (6.74)) are two different parameterizations of the same aggregate forward market demand function. If we assume that there exists a function $e_\eta^f(\bullet)$ of the signal vector η_R such that $e_\eta^f(\eta_R)=\varepsilon_0^f$ for all relevant η_R, then we will have that

$$D^f\left(p^f,\varepsilon_0^f\right)=D^f\left(p^f,e_\eta^f(\eta_R)\right)=q_R^{f*}\left(p^f,\eta_R\right). \tag{6.77}$$

We may combine eqs. (6.74)–(6.77) to write

$$
\begin{aligned}
D^f\left(p^f,\varepsilon_0^f\right) &= D^f\left(p^f,e_\eta^f\left(\eta_R\right)\right) \\
&= D_0^f\left(p^f\right)+e_\eta^f\left(\eta_R\right) \\
&= q_R^{f*}\left(p^f,\eta_R\right) \\
&= -\frac{1}{\lambda_R\sigma_{v_R}^2\omega_a}\cdot\left\{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\left[\phi_1\bar{S}_1^f\left(p^f\right)+\phi_2\bar{S}_2^f\left(p^f\right)\right]+\frac{p^f}{\omega_a}\right\} \\
&\quad +\frac{1}{\lambda_R\sigma_{v_R}^2\omega_a}\cdot\left\{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\eta_R+\bar{v}_R+\frac{\lambda_R\sigma_{v_R^2,v_R}\omega_a}{2}\cdot\left(2-\gamma^s\omega_a\right)\right. \\
&\qquad\qquad \left.+\omega_b\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\right\} \\
&= -\frac{1}{\lambda_R\sigma_{v_R}^2\omega_a}\cdot\left\{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\right. \\
&\qquad\qquad \cdot\left[\phi_1\left[\bar{S}_1^f\left(p^f\right)-\bar{S}_1^f\left(p_0^f\right)\right]+\phi_2\left[\bar{S}_2^f\left(p^f\right)-\bar{S}_2^f\left(p_0^f\right)\right]\right] \\
&\qquad\qquad \left.+\frac{p^f-p_0^f}{\omega_a}\right\} \\
&\quad +\frac{1}{\lambda_R\sigma_{v_R}^2\omega_a}\cdot\left\{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\eta_R+\bar{v}_R\right. \\
&\qquad\qquad +\frac{\lambda_R\sigma_{v_R^2,v_R}\omega_a}{2}\cdot\left(2-\gamma^s\omega_a\right)-\frac{p_0^f}{\omega_a} \\
&\qquad\qquad \left.+\left[\omega_b-\phi_1\bar{S}_1^f\left(p_0^f\right)-\phi_2\bar{S}_2^f\left(p_0^f\right)\right]\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\right\},
\end{aligned}
$$

where we recall from subsection 3.1.10 that p_0^f is an arbitrary reference price in the interval $\left[\underset{\smile}{p}^f,\widehat{p}^f\right]$ over which we defined the forward market demand function. Comparing the second and fifth equalities above, we have that

$$
\begin{aligned}
D_0^f\left(p^f\right) = -\frac{1}{\lambda_R\sigma_{v_R}^2\omega_a}\cdot&\left\{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\right. \\
&\cdot\left[\phi_1\left[\bar{S}_1^f\left(p^f\right)-\bar{S}_1^f\left(p_0^f\right)\right]+\phi_2\left[\bar{S}_2^f\left(p^f\right)-\bar{S}_2^f\left(p_0^f\right)\right]\right] \\
&\left.+\frac{p^f-p_0^f}{\omega_a}\right\}.
\end{aligned} \tag{6.78}
$$

Moreover, we may confirm that the function $e_\eta^f(\eta_R)=\varepsilon_0^f$ indeed exists, and in particular,

$$
\begin{aligned}
e_\eta^f(\eta_R) &= \varepsilon_0^f \\
&= \frac{1}{\lambda_R\sigma_{v_R}^2\omega_a}\cdot\Bigg\{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\eta_R+\bar{v}_R+\frac{\lambda_R\sigma_{v_R^2,v_R}\omega_a}{2}\cdot\left(2-\gamma^s\omega_a\right)-\frac{p_0^f}{\omega_a} \\
&\qquad +\left[\omega_b-\phi_1\bar{S}_1^f\left(p_0^f\right)-\phi_2\bar{S}_2^f\left(p_0^f\right)\right]\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\Bigg\}.
\end{aligned}
\tag{6.79}
$$

Equations (6.78) and (6.79) decompose the forward market demand function $D^f\left(p^f, e_\eta^f(\eta_R)\right)$ into

1. the *price-dependent* shape component $D_0^f\left(p^f\right)$ and
2. the *price-independent* stochastic shock $\varepsilon_0^f=e_\eta^f(\eta_R)$ of the forward market demand function.

Note that $D_0^f\left(p^f\right)$ in eq. (6.78) is a deterministic function of p^f; using this equation, we may verify that $D_0^f\left(p_0^f\right)=0$. Equation (6.79) indicates that $e_\eta^f(\eta_R)$ depends on the realizations of the signals η_R and the expectations $\bar{v}_R$, but not on the realizations v_R, since v_R is revealed after the forward market clears. In addition, it is possible to show in eqs. (6.78) and (6.79) that $\left(2-\gamma^s\omega_a\right)>0$ and $\left(1-\gamma^s\omega_a\right)>0$ for all permissible parameter values.

We may further decompose the expression for ε_0^f in (6.79) into a *stochastic component*

$$\left[\frac{1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)}{\lambda_R\sigma_{v_R}^2\omega_a}\right]\eta_R, \tag{6.80}$$

and a *deterministic component*

$$\frac{1}{\lambda_R \sigma_{v_R}^2 \omega_a} \cdot \left\{ \bar{v}_R + \frac{\lambda_R \sigma_{v_R^2, v_R} \omega_a}{2} \cdot \left(2 - \gamma^s \omega_a\right) - \frac{p_0^f}{\omega_a} \right. \\ \left. + \left[\omega_b - \phi_1 \bar{S}_1^f \left(p_0^f\right) - \phi_2 \bar{S}_2^f \left(p_0^f\right) \right] \left[1 + \lambda_R \sigma_{v_R}^2 \omega_a \left(1 - \gamma^s \omega_a\right) \right] \right\}. \tag{6.81}$$

Note that the expression (6.80) collects the factors dependent on η_R in eq. (6.79) for $e_\eta^f\left(\eta_R\right)$, so that we may write $e_\eta^{f\prime}\left(\eta_R\right)$ as

$$e_\eta^{f\prime}\left(\eta_R\right) = \frac{1 + \lambda_R \sigma_{v_R}^2 \omega_a \left(2 - \gamma^s \omega_a\right)}{\lambda_R \sigma_{v_R}^2 \omega_a} > 0\,. \tag{6.82}$$

Finally, consider how the signal η_R affects the level of forward market demand. Applying the chain rule to the function $D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)$, we have that

$$\frac{\partial D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)}{\partial \eta_R} = \frac{\partial D^f\left(p^f, \varepsilon_0^f\right)}{\partial \varepsilon_0^f} \cdot \frac{d e_\eta^f\left(\eta_R\right)}{d\eta_R}\,.$$

Using eqs. (6.76) and (6.82), we may conclude from the above equation that

$$\frac{\partial D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)}{\partial \eta_R} = \frac{d e_\eta^f\left(\eta_R\right)}{d\eta_R} > 0\,. \tag{6.83}$$

The inequality (6.83) indicates that an increase in *R*'s signal η_R shifts $D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)$ to the right.

6.7 The relationship of demand shocks and prices across markets

This section revisits an expression from the previous chapter (eq. (5.36)) for $d\,\mathrm{E}\left(p^s \middle| p^f\right) \big/ dp^f$, which we established as a function of the derivative

$d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right) \big/ d\varepsilon_0^f$.[219] In subsection 6.7.1, we derive expressions for $d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right) \big/ d\varepsilon_0^f$, and likewise for $d\,\mathrm{E}\left(p^s \middle| p^f\right) \big/ dp^f$ in subsection 6.7.2.

6.7.1 *The derivative* $d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right) \big/ d\varepsilon_0^f$

Begin by taking conditional expectations of ε^s, conditional on ε_0^f, from eq. (6.55):

$$\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right) = \mathrm{E}\left(\eta_R \middle| \varepsilon_0^f\right) + \mathrm{E}\left(\nu_R \middle| \varepsilon_0^f\right).$$

This equation is an identity in ε_0^f so that we may differentiate it with respect to ε_0^f to obtain

$$\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} = \frac{d\,\mathrm{E}\left(\eta_R \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} + \frac{d\,\mathrm{E}\left(\nu_R \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f}. \tag{6.84}$$

Since ν_R is exogenous, we have that $\mathrm{E}\left(\nu_R \middle| \varepsilon_0^f\right) = \mathrm{E}\left(\nu_R\right)$, and hence

$$\frac{d\,\mathrm{E}\left(\nu_R \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} = \frac{d\,\mathrm{E}\left(\nu_R\right)}{d\varepsilon_0^f} = 0.$$

Using this result, eq. (6.84) becomes simply

$$\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} = \frac{d\,\mathrm{E}\left(\eta_R \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f}. \tag{6.85}$$

To find $d\,\mathrm{E}\left(\eta_R \middle| \varepsilon_0^f\right) \big/ d\varepsilon_0^f$, we may solve the second equation in (6.79) for η_R in terms of ε_0^f to obtain

---

[219] We may interpret this derivative as the effect of forward market (public) information on expectations concerning the level of spot market demand.

$$\eta_R = \frac{1}{1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)}\cdot\Bigg\{\lambda_R\sigma_{v_R}^2\omega_a\varepsilon_0^f-\bar{v}_R-\frac{\lambda_R\sigma_{v_R^2,v_R}\omega_a}{2}\cdot\left(2-\gamma^s\omega_a\right)+\frac{p_0^f}{\omega_a}$$
$$-\left[\omega_b-\phi_1\bar{S}_1^f\left(p_0^f\right)-\phi_2\bar{S}_2^f\left(p_0^f\right)\right]$$
$$\cdot\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\Bigg\}. \tag{6.86}$$

Taking conditional expectations of this equation and differentiating with respect to ε_0^f, we obtain

$$\frac{d\,\mathrm{E}\left(\eta_R\middle|\varepsilon_0^f\right)}{d\varepsilon_0^f}=\frac{\lambda_R\sigma_{v_R}^2\omega_a}{1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)}. \tag{6.87}$$

Substituting from eq. (6.87) into eq. (6.85), we also have that

$$\frac{d\,\mathrm{E}\left(\varepsilon^s\middle|\varepsilon_0^f\right)}{d\varepsilon_0^f}=\frac{\lambda_R\sigma_{v_R}^2\omega_a}{1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)}>0. \tag{6.88}$$

As eq. (6.88) indicates, $d\,\mathrm{E}\left(\varepsilon^s\middle|\varepsilon_0^f\right)\Big/d\varepsilon_0^f$ is *constant* as ε_0^f varies (all else equal), given our assumptions. More specifically, $d\,\mathrm{E}\left(\varepsilon^s\middle|\varepsilon_0^f\right)\Big/d\varepsilon_0^f$ is a function of the representative consumer's attributes, the variance of the underlying stochastic parameter v_R, and spot market demand and cost parameters. Comparing eq. (6.88) with eq. (6.82), and noting (from inequality (6.82)) that we may invert $e_\eta^{f\prime}\left(\eta_R\right)$, we see that

$$\frac{d\,\mathrm{E}\left(\varepsilon^s\middle|\varepsilon_0^f\right)}{d\varepsilon_0^f}=\frac{1}{e_\eta^{f\prime}\left(\eta_R\right)}.$$

6.7.2 The derivative $d\,\mathrm{E}\left(p^s \middle| p^f\right) / dp^f$

Using previous results in this chapter, we may simplify the expression for the derivative $d\,\mathrm{E}\left(p^s \middle| p^f\right) / dp^f$. Equation (5.36) first gave an expression for this derivative, which we rewrite below as eq. (6.89):

$$\begin{aligned}\frac{d\,\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f} = \omega_a \Bigg\{ & \left[\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} - \phi_1\right] \bar{S}_1^{f\prime}\left(p^f\right) + \left[\frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} - \phi_2\right] \bar{S}_2^{f\prime}\left(p^f\right) \\ & - \frac{d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right)}{d\varepsilon_0^f} \cdot D_0^{f\prime}\left(p^f\right) \Bigg\}. \end{aligned} \tag{6.89}$$

Next, taking the derivative of $D_0^f\left(p^f\right)$ from eq. (6.78) with respect to p^f, we obtain an expression for $D_0^{f\prime}\left(p^f\right)$:

$$D_0^{f\prime}\left(p^f\right) = -\frac{\left[1 + \lambda_R \sigma_{v_R}^2 \omega_a \left(1 - \gamma^s \omega_a\right)\right]\left[\phi_1 \bar{S}_1^{f\prime}\left(p^f\right) + \phi_2 \bar{S}_2^{f\prime}\left(p^f\right)\right] + \dfrac{1}{\omega_a}}{\lambda_R \sigma_{v_R}^2 \omega_a}. \tag{6.90}$$

Using eqs. (6.88) and (6.90) to substitute for $d\,\mathrm{E}\left(\varepsilon^s \middle| \varepsilon_0^f\right) / d\varepsilon_0^f$ and $D_0^{f\prime}\left(p^f\right)$, respectively, in eq. (6.89) yields

$$
\begin{aligned}
\frac{d\,\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f} = \omega_a \Bigg\{ & \left[\frac{\lambda_R \sigma_{v_R}^2 \omega_a}{1+\lambda_R \sigma_{v_R}^2 \omega_a \left(2-\gamma^s \omega_a\right)} - \phi_1 \right] \bar{S}_1^{f\prime}\left(p^f\right) \\
& + \left[\frac{\lambda_R \sigma_{v_R}^2 \omega_a}{1+\lambda_R \sigma_{v_R}^2 \omega_a \left(2-\gamma^s \omega_a\right)} - \phi_2 \right] \bar{S}_2^{f\prime}\left(p^f\right) \\
& - \frac{\lambda_R \sigma_{v_R}^2 \omega_a}{1+\lambda_R \sigma_{v_R}^2 \omega_a \left(2-\gamma^s \omega_a\right)} \\
& \cdot \left[- \frac{\left[1+\lambda_R \sigma_{v_R}^2 \omega_a \left(1-\gamma^s \omega_a\right)\right]\left[\phi_1 \bar{S}_1^{f\prime}\left(p^f\right) + \phi_2 \bar{S}_2^{f\prime}\left(p^f\right)\right] + \frac{1}{\omega_a}}{\lambda_R \sigma_{v_R}^2 \omega_a} \right] \Bigg\}.
\end{aligned}
$$

Collecting terms and simplifying, this becomes

$$
\frac{d\,\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f} = \frac{\lambda_R \sigma_{v_R}^2 \omega_a^2 \left[\left(1-\phi_1\right) \bar{S}_1^{f\prime}\left(p^f\right) + \left(1-\phi_2\right) \bar{S}_2^{f\prime}\left(p^f\right)\right] + 1}{1+\lambda_R \sigma_{v_R}^2 \omega_a \left(2-\gamma^s \omega_a\right)}. \tag{6.91}
$$

Assuming strictly increasing forward market SFs for all p^f, $\bar{S}_i^{f\prime}\left(p^f\right) > 0$, we conclude that

$$
\frac{d\,\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f} > 0. \tag{6.92}
$$

6.8 Properties of aggregate forward market demand $D^f\left(p^f, \varepsilon_0^f\right)$

The final section of this chapter summarizes the salient properties of $D_0^f\left(p^f\right)$, ε_0^f, and their sum, aggregate forward market demand $D^f\left(p^f, \varepsilon_0^f\right) = D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)$ $= D_0^f\left(p^f\right) + \varepsilon_0^f$ based on eqs. (6.76)–(6.79) in section 6.6.

6.8.1 *Properties of* $D_0^f\left(p^f\right)$

Given our parametric assumptions—and once we have determined the forward market SFs $\bar{S}_i^f\left(p^f\right)$ (see chapter 7)—the shape component $D_0^f\left(p^f\right)$ of forward market demand (see eq. (6.78)) is *deterministic*, *differentiable*, and *common knowledge*. Signing the result of eq. (6.90) from the previous section (again assuming $\bar{S}_i^{f\prime}\left(p^f\right)>0$, $i=1,2$), we get

$$D_0^{f\prime}\left(p^f\right)=-\frac{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\left[\phi_1\bar{S}_1^{f\prime}\left(p^f\right)+\phi_2\bar{S}_2^{f\prime}\left(p^f\right)\right]+\dfrac{1}{\omega_a}}{\lambda_R\sigma_{v_R}^2\omega_a}<0. \quad (6.93)$$

From eq. (6.93) we conclude, under our parametric assumptions and assuming strictly increasing forward market SFs, that $D_0^f\left(p^f\right)$ and hence $D^f\left(p^f,\varepsilon_0^f\right)$ are *downward-sloping* in p^f.[220]

6.8.2 *Properties of* ε_0^f

Given our parametric assumptions, the shock ε_0^f in eq. (6.79) is a function of *R*'s exogenous signal η_R, and includes both *stochastic* and *deterministic* components as indicated in the discussion of that equation. In addition, the stochastic component of ε_0^f has a stationary, common knowledge distribution $F_{\varepsilon_0^f}\left(\varepsilon_0^f\right)$, since the distribution $F_{\eta_R}\left(\eta_R\right)$ has these properties.

---

[220] Moreover, $D_0^f\left(p^f\right)$ would be affine in p^f if the functions $\bar{S}_i^f\left(p^f\right)$ are affine for firms $i=1, 2$, though we do not impose this affine restriction here.

Now consider the support E^f of ε_0^f. In subsection 6.2.1, we assumed that the support of any consumer's signal—and therefore of η_R—was $\mathbb{R}_+$. Recalling that $\varepsilon_0^f = e_\eta^f\left(\eta_R\right)$ from eq. (6.79), we may obtain an expression for the *minimum value* of the forward market demand shock—denoted as $\underset{\smile}{\varepsilon}_0^f$—by substituting $\eta_R = 0$ into eq. (6.79). Doing so yields

$$\begin{aligned} \underset{\smile}{\varepsilon}_0^f &= e_\eta^f\left(0\right) \\ &= \frac{1}{\lambda_R \sigma_{\nu_R}^2 \omega_a} \cdot \left\{ \overline{\nu}_R + \frac{\lambda_R \sigma_{\nu_R^2,\nu_R} \omega_a}{2} \cdot \left(2 - \gamma^s \omega_a\right) - \frac{p_0^f}{\omega_a} \right. \\ &\qquad \left. + \left[\omega_b - \phi_1 \overline{S}_1^f\left(p_0^f\right) - \phi_2 \overline{S}_2^f\left(p_0^f\right)\right]\left[1 + \lambda_R \sigma_{\nu_R}^2 \omega_a \left(1 - \gamma^s \omega_a\right)\right] \right\}. \end{aligned} \tag{6.94}$$

In other words, ε_0^f is bounded below by some $\underset{\smile}{\varepsilon}_0^f \in \mathbb{R}$ from eq. (6.94) in every round of the market. In terms of the support E^f, we have that

$$\varepsilon_0^f \in E^f \equiv \left[\underset{\smile}{\varepsilon}_0^f, \widehat{\varepsilon}_0^f\right], \tag{6.95}$$

where $\underset{\smile}{\varepsilon}_0^f < \widehat{\varepsilon}_0^f$ is given by eq. (6.94). The support E^f in eq. (6.95) determines the extent of the forward market SFs, that is, the price domain over which they are defined. We need not specify the upper limit $\widehat{\varepsilon}_0^f$ of this support; from eq. (6.79) and given $\eta_R \in \mathbb{R}_+$, $\widehat{\varepsilon}_0^f$ may in principle be infinite.[221] If $\underset{\smile}{\varepsilon}_0^f$ is sufficiently small (and from eq.

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[221] In practice, however, as the next chapter discusses, we will compute forward market SF trajectories over a *finite interval* of prices $\left[\underset{\smile}{p}^f, \widehat{p}^f\right]$, where $\widehat{p}^f = p^{f*}\left(\widehat{\varepsilon}_0^f\right)$, $\widehat{\varepsilon}_0^f = e_\eta^f\left(\widehat{\eta}_R\right)$, and the support of η_R is $\left[0, \widehat{\eta}_R\right]$, $0 < \widehat{\eta}_R < \infty$.

(6.94), we can have $\varepsilon_0^f < 0$), suppliers' forward market quantities will be negative, indicating purchases rather than sales in the forward market.[222]

6.8.3 Properties of $D^f\left(p^f, \varepsilon_0^f\right) = D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)$

The additively separable functional form $D^f\left(p^f, \varepsilon_0^f\right) = D_0^f\left(p^f\right) + \varepsilon_0^f$ assumed in subsection 3.1.10 and restated in eq. (6.76) above implies that

$$\frac{\partial^2 D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)}{\partial p^f \partial \eta_R} = \frac{\partial^2 D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)}{\partial p^f \partial \varepsilon_0^f} \cdot \frac{de_\eta^f\left(\eta_R\right)}{d\eta_R} = 0\,, \tag{6.96}$$

since

$$\frac{\partial^2 D^f\left(p^f, e_\eta^f\left(\eta_R\right)\right)}{\partial p^f \partial \varepsilon_0^f} = \frac{\partial^2 D^f\left(p^f, \varepsilon_0^f\right)}{\partial p^f \partial \varepsilon_0^f} = 0\,.$$

The interpretation of eq. (6.96) is that the signal η_R shifts the forward market demand function horizontally but does not change this function's shape.

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[222] There is at least anecdotal empirical evidence from electricity markets (see, e.g., Federal Energy Regulatory Commission 2003a) that generating firms do frequently take long positions in the forward market. In addition, recent theoretical work (e.g., Hughes and Kao 1997, 128; Pirrong 2000, 15) has suggested that, under a variety of circumstances, such behavior can indeed be profitable.

As we see in the specific numerical examples of chapter 7, however, focusing on *strictly increasing* forward market SFs over reasonable price ranges tends to yield positive forward market quantities (short positions on the part of suppliers) within the present model.

> Equilibrium has become a kind of holy sacrament in economics and has seriously diverted attention from the real world of Heraclitean flux. . . . The economic system is a structure in space-time. Consequently, it is evolutionary, subject to constant and irreversible change.
>
> —Kenneth Boulding

> God does not care about our mathematical difficulties. He integrates empirically.
>
> —Einstein

7 The forward market supply functions in the simplified affine example

WE RETURN IN THIS CHAPTER to the supply-side analysis of chapter 5. Section 7.1 below simplifies further firms' equilibrium optimality conditions for the forward market, while section 7.2 explores the existence and uniqueness properties of solutions to the resulting system of equations and the effect of singularities. Next, in section 7.3, we discuss two complementary numerical strategies for solving this system. We develop qualitative insights into the phase space of solutions in section 7.4 with the help of numerous graphical illustrations. Section 7.5 then describes how we chose values of certain model parameters to enhance the verisimilitude of the model. Section 7.6 presents an equilibrium selection procedure and conducts comparative statics analysis to investigate the effects of parameter variations on firms' forward market SFs. To conclude the chapter, section 7.7 compares expected welfare under the multi-settlement SFE model with that under alternative behavioral assumptions and market architectures.

7.1 Equilibrium optimality conditions for the forward market

7.1.1 Integrating previous chapters' results concerning the functions $\mathrm{E}\left(p^s \middle| p^f\right)$ *and* $D_0^{f\prime}\left(p^f\right)$

Begin by recalling eq. (5.37), firm 1's equilibrium optimality condition for the forward market under the assumptions of the simplified affine example, rewritten as eq. (7.1) below:

$$\begin{aligned}\Big\{\phi_1\phi_2\Big[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{01}+c_1\bar{S}_1^f\left(p^f\right)\right)\Big]-\Big[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\Big]\Big\}\bar{S}_2^{f\prime}\left(p^f\right)\\ =\bar{S}_1^f\left(p^f\right)-D_0^{f\prime}\left(p^f\right)\Big[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\Big].\end{aligned} \tag{7.1}$$

The analogous condition for firm 2 is, by symmetry,

$$\begin{aligned}\Big\{\phi_1\phi_2\Big[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{02}+c_2\bar{S}_2^f\left(p^f\right)\right)\Big]-\Big[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\Big]\Big\}\bar{S}_1^{f\prime}\left(p^f\right)\\ =\bar{S}_2^f\left(p^f\right)-D_0^{f\prime}\left(p^f\right)\Big[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\Big].\end{aligned} \tag{7.2}$$

In the following, we use analytical results from chapters 5 and 6 to substitute for the functions $\mathrm{E}\left(p^s\middle|p^f\right)$ and $D_0^{f\prime}\left(p^f\right)$ in eqs. (7.1) and (7.2).

Recall eq. (5.33) for $\mathrm{E}\left(p^s\middle|p^f\right)$ (rewritten as eq. (7.3) below),

$$\mathrm{E}\left(p^s\middle|p^f\right)=\omega_a\Big[\mathrm{E}\left(\varepsilon^s\middle|e_p^f\left(p^f\right)\right)-\phi_1\bar{S}_1^f\left(p^f\right)-\phi_2\bar{S}_2^f\left(p^f\right)+\omega_b\Big], \tag{7.3}$$

where, also in chapter 5, we defined $\omega_a = \left(\beta_1^s + \beta_2^s + \gamma^s\right)^{-1}$ and $\omega_b = c_{01}\beta_1^s + c_{02}\beta_2^s$. To evaluate eq. (7.3) in terms of known constants and functions of p^f, we need to evaluate the expectation $\mathrm{E}\left(\varepsilon^s \middle| e_p^f\left(p^f\right)\right)$. We may do so by appealing to various results from chapter 6, as described below.

Begin with section 6.5's simple model for ε^s (eq. (6.55)),

$$\varepsilon^s = \eta_R + \nu_R. \tag{7.4}$$

Taking expectations of eq. (7.4) conditional on $e_p^f\left(p^f\right) = \varepsilon_0^f$ yields

$$\mathrm{E}\left(\varepsilon^s \middle| e_p^f\left(p^f\right)\right) = \mathrm{E}\left(\eta_R \middle| e_p^f\left(p^f\right)\right) + \mathrm{E}\left(\nu_R \middle| e_p^f\left(p^f\right)\right),$$

which, since ν_R is exogenous, is simply

$$\mathrm{E}\left(\varepsilon^s \middle| e_p^f\left(p^f\right)\right) = \mathrm{E}\left(\eta_R \middle| e_p^f\left(p^f\right)\right) + \bar{\nu}_R. \tag{7.5}$$

Similarly, taking expectations of eq. (6.86) conditional on $e_p^f\left(p^f\right) = \varepsilon_0^f$ gives us

$$\begin{aligned}
&\mathrm{E}\left(\eta_R \middle| e_p^f\left(p^f\right)\right) \\
&\quad = \frac{1}{1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(2-\gamma^s\omega_a\right)} \cdot \Bigg\{ \lambda_R\sigma_{\nu_R}^2\omega_a e_p^f\left(p^f\right) - \bar{\nu}_R - \frac{\lambda_R\sigma_{\nu_R^2,\nu_R}\omega_a}{2}\cdot\left(2-\gamma^s\omega_a\right) \\
&\qquad\qquad + \frac{p_0^f}{\omega_a} - \left[\omega_b - \phi_1\bar{S}_1^f\left(p_0^f\right) - \phi_2\bar{S}_2^f\left(p_0^f\right)\right] \\
&\qquad\qquad\qquad \cdot\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\Bigg\}.
\end{aligned} \tag{7.6}$$

Solving the market-clearing condition for the forward market (eq. (5.29)) for $\varepsilon_0^f = e_p^f\left(p^f\right)$ and substituting p^f for $p^{f*}\left(\varepsilon_0^f\right)$, we have

$$e_p^f\left(p^f\right)=\bar{S}_1^f\left(p^f\right)+\bar{S}_2^f\left(p^f\right)-D_0^f\left(p^f\right). \tag{7.7}$$

Finally, the shape component of forward market demand, $D_0^f\left(p^f\right)$, is (from eq. (6.78))

$$\begin{aligned} D_0^f\left(p^f\right)=-\frac{1}{\lambda_R\sigma_{\nu_R}^2\omega_a}\cdot\Big\{&\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\\ &\cdot\left[\phi_1\left[\bar{S}_1^f\left(p^f\right)-\bar{S}_1^f\left(p_0^f\right)\right]+\phi_2\left[\bar{S}_2^f\left(p^f\right)-\bar{S}_2^f\left(p_0^f\right)\right]\right]\\ &+\frac{p^f-p_0^f}{\omega_a}\Big\}. \end{aligned} \tag{7.8}$$

Combining eqs. (7.5)–(7.8) to simplify eq. (7.3) and collecting terms yields the desired result,[223]

$$\begin{aligned} &\mathrm{E}\left(p^s\middle|p^f\right)\\ &=\frac{\lambda_R\sigma_{\nu_R}^2\omega_a^2\left[\left(1-\phi_1\right)\bar{S}_1^f\left(p^f\right)+\left(1-\phi_2\right)\bar{S}_2^f\left(p^f\right)+\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{\nu}_R-\dfrac{\sigma_{\nu_R^2,\nu_R}}{2\sigma_{\nu_R}^2}\right)\right]+p^f}{1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(2-\gamma^s\omega_a\right)}. \end{aligned} \tag{7.9}$$

We turn next to the function $D_0^{f\prime}\left(p^f\right)$, the slope of the shape component of forward market demand. Differentiating eq. (7.8) with respect to p^f, we have that

---

[223] The derivative of eq. (7.9) is consistent with $d\mathrm{E}\left(p^s\middle|p^f\right)\big/dp^f$ from eq. (6.91). Note also that $\mathrm{E}\left(p^s\middle|p^f\right)$ in eq. (7.9) (and ultimately, the forward market supply and demand functions) depend only on three moments of ν_R — $\bar{\nu}_R$, $\sigma_{\nu_R}^2$, and $\sigma_{\nu_R^2,\nu_R}$ —rather than on ν_R's entire distribution. For computational purposes, we assume in Appendix F.1.5 that ν_R is lognormally distributed, which permits us to express $\sigma_{\nu_R^2,\nu_R}$ as a function of the other two moments. We choose the parameters $\bar{\nu}_R$ and $\sigma_{\nu_R}^2$, in turn, via an empirically-based benchmarking procedure described in section 7.5.

$$D_0^{f\prime}\left(p^f\right) = -\frac{\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\left[\phi_1\bar{S}_1^{f\prime}\left(p^f\right)+\phi_2\bar{S}_2^{f\prime}\left(p^f\right)\right]+\dfrac{1}{\omega_a}}{\lambda_R\sigma_{v_R}^2\omega_a}. \qquad (7.10)$$

The three equations (7.1), (7.2), and (7.10) constitute a system of nonlinear ordinary differential equations (ODEs) implicitly characterizing the forward market SFs $\bar{S}_1^f\left(p^f\right)$ and $\bar{S}_2^f\left(p^f\right)$ as well as the slope of forward market demand, $D_0^{f\prime}\left(p^f\right)$, where eq. (7.9) gives an expression for $\mathrm{E}\left(p^s \middle| p^f\right)$. Each consumer[224] solves her forward market optimization problem (as in chapter 6) given the two SFs $\bar{S}_i^f\left(p^f\right)$, and given an equilibrium price in both the forward and spot markets. Each supplier i ($i, j = 1, 2,\ i \neq j$) maximizes its profits, taking supplier j's SF as given (the Nash assumption), and also taking consumers' actions as given.

Each equation in the system (7.1), (7.2), and (7.10) arises from the respective optimization problems of the duopoly suppliers and the representative consumer. In order to solve this system numerically using commercially-available differential equation solvers, however, we have found it useful to rearrange this three-equation system by isolating the derivatives of the dependent variables $\bar{S}_1^{f\prime}\left(p^f\right)$ and $\bar{S}_2^{f\prime}\left(p^f\right)$.[225] In addition, the simplifications we undertake in the remainder of this section are useful in highlighting certain quadratic forms that characterize several loci of interest, as detailed in section 7.4 below.

[224] Recall that eq. (7.10) above expresses the slope of aggregate forward market demand, the sum of individual consumers' forward market demand functions.

[225] We do this in subsection 7.2.2 for a restricted version of this three-equation system. While the resulting expressions appear, if anything, more complicated than the original system, the revised system does have the virtue of isolating the vector of supply functions' derivatives.

As a first step toward solving the system (7.1), (7.2), and (7.10), we may reduce these three equations to a two-equation system in $\bar{S}_1^f\left(p^f\right)$ and $\bar{S}_2^f\left(p^f\right)$ and eliminate $\mathrm{E}\left(p^s\middle|p^f\right)$ by using eqs. (7.9) and (7.10) to substitute for $\mathrm{E}\left(p^s\middle|p^f\right)$ and $D_0^{f\prime}\left(p^f\right)$ in eqs. (7.1) and (7.2). Making these substitutions and collecting terms yields the two equations

$$
\begin{aligned}
&\phi_1\omega_a\Big[1+\lambda_R\sigma^2_{\nu_R}\omega_a\left(1-\gamma^s\omega_a\right)\Big]\Bigg\{\left(1-\phi_1\right)\bar{S}_1^f\left(p^f\right)+\left(1-\phi_2\right)\bar{S}_2^f\left(p^f\right)-\frac{2-\gamma^s\omega_a}{\omega_a}\cdot p^f\\
&\qquad\qquad\qquad+\Bigg[\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{\nu}_R-\frac{\sigma_{\nu_R^2,\nu_R}}{2\sigma^2_{\nu_R}}\right)\Bigg]\Bigg\}\bar{S}_1^{f\prime}\left(p^f\right)\\
&+\Bigg[\bigg(c_1\phi_1\phi_2\Big[1+\lambda_R\sigma^2_{\nu_R}\omega_a\left(2-\gamma^s\omega_a\right)\Big]\\
&\quad+\omega_a\left(1-\phi_1\right)\Big\{\lambda_R\sigma^2_{\nu_R}\omega_a\Big[\left(1-\phi_1\phi_2\right)+\phi_2\left(1-\gamma^s\omega_a\right)\Big]+\phi_2\Big\}\bigg)\bar{S}_1^f\left(p^f\right)\\
&\quad+\omega_a\left(1-\phi_2\right)\Big\{\lambda_R\sigma^2_{\nu_R}\omega_a\Big[\left(1-\phi_1\phi_2\right)+\phi_2\left(1-\gamma^s\omega_a\right)\Big]+\phi_2\Big\}\bar{S}_2^f\left(p^f\right)\\
&\quad-\Big[\left(2-\gamma^s\omega_a\right)\Big\{\lambda_R\sigma^2_{\nu_R}\omega_a\Big[1+\phi_2\left(1-\gamma^s\omega_a\right)\Big]+\phi_2\Big\}+\phi_1\phi_2\Big]p^f\\
&\quad+\Bigg(c_{01}\phi_1\phi_2\Big[1+\lambda_R\sigma^2_{\nu_R}\omega_a\left(2-\gamma^s\omega_a\right)\Big]+\omega_a\Bigg[\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{\nu}_R-\frac{\sigma_{\nu_R^2,\nu_R}}{2\sigma^2_{\nu_R}}\right)\Bigg]\\
&\qquad\qquad\qquad\cdot\Big\{\lambda_R\sigma^2_{\nu_R}\omega_a\Big[\left(1-\phi_1\phi_2\right)+\phi_2\left(1-\gamma^s\omega_a\right)\Big]+\phi_2\Big\}\Bigg)\Bigg]\bar{S}_2^{f\prime}\left(p^f\right)\\
&+\Bigg(\Big\{\left(1-\phi_1\right)+\Big[1+\lambda_R\sigma^2_{\nu_R}\omega_a\left(2-\gamma^s\omega_a\right)\Big]\Big\}\bar{S}_1^f\left(p^f\right)+\left(1-\phi_2\right)\bar{S}_2^f\left(p^f\right)\\
&\quad-\frac{2-\gamma^s\omega_a}{\omega_a}\cdot p^f+\Bigg[\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{\nu}_R-\frac{\sigma_{\nu_R^2,\nu_R}}{2\sigma^2_{\nu_R}}\right)\Bigg]\Bigg)=0
\end{aligned}
\tag{7.11}
$$

and

$$
\begin{aligned}
&\Big[\omega_a(1-\phi_1)\Big\{\lambda_R\sigma^2_{v_R}\omega_a\Big[(1-\phi_1\phi_2)+\phi_1\big(1-\gamma^s\omega_a\big)\Big]+\phi_1\Big\}\bar{S}^f_1\big(p^f\big)\\
&\quad+\Big(c_2\phi_1\phi_2\Big[1+\lambda_R\sigma^2_{v_R}\omega_a\big(2-\gamma^s\omega_a\big)\Big]\\
&\qquad+\omega_a(1-\phi_2)\Big\{\lambda_R\sigma^2_{v_R}\omega_a\Big[(1-\phi_1\phi_2)+\phi_1\big(1-\gamma^s\omega_a\big)\Big]+\phi_1\Big\}\Big)\bar{S}^f_2\big(p^f\big)\\
&\quad-\Big[\big(2-\gamma^s\omega_a\big)\Big\{\lambda_R\sigma^2_{v_R}\omega_a\Big[1+\phi_1\big(1-\gamma^s\omega_a\big)\Big]+\phi_1\Big\}+\phi_1\phi_2\Big]p^f\\
&\quad+\left(c_{02}\phi_1\phi_2\Big[1+\lambda_R\sigma^2_{v_R}\omega_a\big(2-\gamma^s\omega_a\big)\Big]+\omega_a\left[\omega_b+\big(2-\gamma^s\omega_a\big)\left(\bar{v}_R-\frac{\sigma_{v_R^2,v_R}}{2\sigma^2_{v_R}}\right)\right]\right.\\
&\qquad\qquad\left.\cdot\Big\{\lambda_R\sigma^2_{v_R}\omega_a\Big[(1-\phi_1\phi_2)+\phi_1\big(1-\gamma^s\omega_a\big)\Big]+\phi_1\Big\}\right)\Big]\bar{S}^{f\prime}_1\big(p^f\big)\\
&+\phi_2\omega_a\Big[1+\lambda_R\sigma^2_{v_R}\omega_a\big(1-\gamma^s\omega_a\big)\Big]\Big\{(1-\phi_1)\bar{S}^f_1\big(p^f\big)+(1-\phi_2)\bar{S}^f_2\big(p^f\big)-\frac{2-\gamma^s\omega_a}{\omega_a}\cdot p^f\\
&\qquad\qquad+\left[\omega_b+\big(2-\gamma^s\omega_a\big)\left(\bar{v}_R-\frac{\sigma_{v_R^2,v_R}}{2\sigma^2_{v_R}}\right)\right]\Big\}\bar{S}^{f\prime}_2\big(p^f\big)\\
&+\Big((1-\phi_1)\bar{S}^f_1\big(p^f\big)+\Big\{(1-\phi_2)+\Big[1+\lambda_R\sigma^2_{v_R}\omega_a\big(2-\gamma^s\omega_a\big)\Big]\Big\}\bar{S}^f_2\big(p^f\big)\\
&\quad-\frac{2-\gamma^s\omega_a}{\omega_a}\cdot p^f+\left[\omega_b+\big(2-\gamma^s\omega_a\big)\left(\bar{v}_R-\frac{\sigma_{v_R^2,v_R}}{2\sigma^2_{v_R}}\right)\right]\Big)=0.
\end{aligned}
\tag{7.12}
$$

In the next subsection, we examine the structure of eqs. (7.11) and (7.12) and recast them in a form more convenient for numerical solution.

7.1.2 The structure of equations (7.11) and (7.12)

To make clear the structure of eqs. (7.11) and (7.12), define some additional notation. First, let a superscript "$^\top$" be the matrix (or vector) transpose operator. Let $\bar{S}^{f++}\big(p^f\big)$, given by

$$
\bar{S}^{f++}\big(p^f\big)\equiv\big(\bar{S}^f_1\big(p^f\big)\quad\bar{S}^f_2\big(p^f\big)\quad p^f\quad 1\big)^\top, \tag{7.13}
$$

be an $(n+2)$ x 1 column vector of supply functions, augmented by the independent variable p^f and the number "1."[226] The derivative of $\overline{S}^{f++}\left(p^f\right)$ with respect to p^f is, from eq. (7.13),

$$\overline{S}^{f++\prime}\left(p^f\right)=\left(\overline{S}_1^{f\prime}\left(p^f\right) \quad \overline{S}_2^{f\prime}\left(p^f\right) \quad 1 \quad 0\right)^{\top}. \tag{7.14}$$

Now define $\mathscr{P}_k^i$, given by (suppressing its dependence on p^f in the following for notational simplicity)

$$\mathscr{P}_k^i \equiv \left(C_k^i\right)^{\top} \overline{S}^{f++}\left(p^f\right), \tag{7.15}$$

as the first-order polynomial in the elements of $\overline{S}^{f++}\left(p^f\right)$ that multiplies the k^{th} component of $\overline{S}^{f++\prime}\left(p^f\right)$ (recall eq. (7.14)) in eqs. (7.11) (for $i=1$) and (7.12) (for $i=2$). In eq. (7.15), define

$$C_k^i \equiv \left(C_{k,1}^i \quad C_{k,2}^i \quad C_{k,3}^i \quad C_{k,4}^i\right)^{\top} \tag{7.16}$$

as an $(n+2)$ x 1 column vector of constant, exogenous coefficients $C_{k,l}^i$ (defined below), with $l=1,2,3,4$ indexing the elements of the vector C_k^i. Writing out the polynomial $\mathscr{P}_k^i$ explicitly, we have

$$\mathscr{P}_k^i \equiv C_{k,1}^i \overline{S}_1^f\left(p^f\right)+C_{k,2}^i \overline{S}_2^f\left(p^f\right)+C_{k,3}^i p^f+C_{k,4}^i. \tag{7.17}$$

---

[226] Hence the superscript "++" in the notation $\overline{S}^{f++}\left(p^f\right)$.

We define each coefficient $C^i_{k,l}$ by comparing the definition of $\mathscr{P}^i_k$ with the respective coefficients in eqs. (7.11) and (7.12).[227] Using the notation $\mathscr{P}^i_k$, we may write eqs. (7.11) and (7.12) more compactly as

$$\mathscr{P}^1_1 \bar{S}^{f\prime}_1\left(p^f\right) + \mathscr{P}^1_2 \bar{S}^{f\prime}_2\left(p^f\right) + \mathscr{P}^1_3 = 0 \tag{7.18}$$

and

$$\mathscr{P}^2_1 \bar{S}^{f\prime}_1\left(p^f\right) + \mathscr{P}^2_2 \bar{S}^{f\prime}_2\left(p^f\right) + \mathscr{P}^2_3 = 0\,. \tag{7.19}$$

7.1.3 *Isolating the $\bar{S}^{f\prime}_i\left(p^f\right)$ in equations (7.18) and (7.19)*

For computational purposes, it is useful to recast eqs. (7.18) and (7.19) so that each derivative $\bar{S}^{f\prime}_i\left(p^f\right)$ appears in only one equation. Doing so yields

$$\left(\mathscr{P}^1_1 \mathscr{P}^2_2 - \mathscr{P}^1_2 \mathscr{P}^2_1\right) \bar{S}^{f\prime}_1\left(p^f\right) = \mathscr{P}^1_2 \mathscr{P}^2_3 - \mathscr{P}^1_3 \mathscr{P}^2_2 \tag{7.20}$$

and

$$\left(\mathscr{P}^1_2 \mathscr{P}^2_1 - \mathscr{P}^1_1 \mathscr{P}^2_2\right) \bar{S}^{f\prime}_2\left(p^f\right) = \mathscr{P}^1_1 \mathscr{P}^2_3 - \mathscr{P}^1_3 \mathscr{P}^2_1\,, \tag{7.21}$$

where we impose the restriction that the determinant of the coefficient matrix in eqs. (7.18) and (7.19) is nonzero, that is,

$$\mathscr{P}^1_1 \mathscr{P}^2_2 - \mathscr{P}^1_2 \mathscr{P}^2_1 \neq 0\,. \tag{7.22}$$

Given the restriction (7.22), the two systems [(7.18), (7.19)] and [(7.20), (7.21)] are

[227] For convenience, Appendix E.1 defines each coefficient $C^i_{k,l}$ explicitly.

equivalent in the sense that the sets of solutions to each of these two systems coincide. Note that the coefficient of $\bar{S}_1^{f\prime}\left(p^f\right)$ in eq. (7.20), $\left(\mathcal{P}_1^1\mathcal{P}_2^2-\mathcal{P}_2^1\mathcal{P}_1^2\right)$, is just the additive inverse of the coefficient of $\bar{S}_2^{f\prime}\left(p^f\right)$ in eq. (7.21), $\left(\mathcal{P}_2^1\mathcal{P}_1^2-\mathcal{P}_1^1\mathcal{P}_2^2\right)$. An implication is that the coefficients $\left(\mathcal{P}_1^1\mathcal{P}_2^2-\mathcal{P}_2^1\mathcal{P}_1^2\right)$ and $\left(\mathcal{P}_2^1\mathcal{P}_1^2-\mathcal{P}_1^1\mathcal{P}_2^2\right)$ vanish over the same set of parameter values. This property will be important in the next section and in Appendix E.2 in characterizing properties of the phase space that the solutions to the system (7.20) and (7.21) inhabit.

Let both j and k index elements of the vector $\bar{S}^{f++\prime}\left(p^f\right)$ in eq. (7.14). By multiplying out the coefficients of the form $\left(\mathcal{P}_j^1\mathcal{P}_k^2-\mathcal{P}_k^1\mathcal{P}_j^2\right)$ in eqs. (7.20) and (7.21), Appendix E.1 makes explicit that these coefficients are *quadratic forms* in the elements of $\bar{S}^{f++}\left(p^f\right)$, as eqs. (7.17), (7.20), and (7.21) imply. Next, let $\mathcal{Q}_{jk}$ be an $\left(n+2\right)$ x $\left(n+2\right)$ symmetric matrix. We define $\mathcal{Q}_{jk}$ implicitly below such that its elements are functions of the coefficients $C_{k,l}^i$. In particular, for a coefficient $\left(\mathcal{P}_j^1\mathcal{P}_k^2-\mathcal{P}_k^1\mathcal{P}_j^2\right)$ in eqs. (7.20) and (7.21), the following relationship defines elements of $\mathcal{Q}_{jk}$ in terms of the coefficients of the polynomials $\mathcal{P}_j^i$ and $\mathcal{P}_k^i$:

$$\bar{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{jk}\bar{S}^{f++}\left(p^f\right)\equiv\mathcal{P}_j^1\mathcal{P}_k^2-\mathcal{P}_k^1\mathcal{P}_j^2\,. \tag{7.23}$$

From definition (7.23), we have that

$$\mathcal{Q}_{21}=-\mathcal{Q}_{12}\,. \tag{7.24}$$

Using the notation of eq. (7.23), we may rewrite eqs. (7.20) and (7.21) as

$$\left[\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{12}\bar{S}^{f++}\left(p^f\right)\right]\bar{S}_1^{f\prime}\left(p^f\right) = \bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{23}\bar{S}^{f++}\left(p^f\right) \tag{7.25}$$

and

$$\left[\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{21}\bar{S}^{f++}\left(p^f\right)\right]\bar{S}_2^{f\prime}\left(p^f\right) = \bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{13}\bar{S}^{f++}\left(p^f\right), \tag{7.26}$$

whereby the condition (7.22) becomes

$$\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{12}\bar{S}^{f++}\left(p^f\right) \neq 0\,. \tag{7.27}$$

Equation (7.24) implies, moreover, that

$$\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{21}\bar{S}^{f++}\left(p^f\right) = -\left[\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{12}\bar{S}^{f++}\left(p^f\right)\right]. \tag{7.28}$$

Appendix E.1 provides explicit expressions for the elements of the matrices $\mathcal{Q}_{jk}$ in eqs. (7.25) and (7.26). Under the restriction (7.27), equations (7.25) and (7.26) constitute a transformed version of the original ODE system (7.11) and (7.12) above characterizing firms' optimal forward market actions. Later in this chapter, we compute numerical solutions of this system for a restricted domain of prices p^f.

The coefficient matrices $\mathcal{Q}_{jk}$ in eqs. (7.25) and (7.26) are functions, ultimately, of parameters characterizing[228] suppliers' marginal costs, stochastic distributions, and consumers' technology, utility and risk preferences. Hence, given values for these primitive parameters, the elements of $\mathcal{Q}_{jk}$ are simply known, exogenous constants. Equations (7.25) and (7.26) constitute a coupled system of first-order nonlinear ODEs in

[228] In part through the intermediate variables γ^s, ϕ_1, ϕ_2, ω_a, and ω_b.

the equilibrium forward market quantities, $\bar{q}_1^f \equiv \bar{S}_1^f\left(p^f\right)$ and $\bar{q}_2^f \equiv \bar{S}_2^f\left(p^f\right)$, with independent variable p^f. The system needs to be augmented by an "initial condition" $\left[\bar{S}_1^f\left(p^{f,0}\right), \bar{S}_2^f\left(p^{f,0}\right), p^{f,0}\right]$ to have a well-defined, unique solution.[229] We use the term *SF trajectory* (or simply, *trajectory*) to denote a curve $\bar{S}^f\left(p^f\right)$ in $\bar{q}_1^f$-$\bar{q}_2^f$-p^f space (i.e., some subset of $\mathbb{R}^3$) passing through some initial condition $\left[\bar{S}_1^f\left(p^{f,0}\right), \bar{S}_2^f\left(p^{f,0}\right), p^{f,0}\right]$ and solving eqs. (7.25) and (7.26) at every point. The *projections* of this SF trajectory into the p^f-$\bar{q}_1^f$ and p^f-$\bar{q}_2^f$ planes, in turn, are identically the SFs $\bar{S}_1^f\left(p^f\right)$ and $\bar{S}_2^f\left(p^f\right)$ for firms 1 and 2. Once we solve for the SFs $\bar{S}_i^f\left(p^f\right)$, we may compute the slope of the shape component of forward market demand, $D_0^{f\prime}\left(p^f\right)$, from eq. (7.10).

While there are no known methods of solving a system of the form of eqs. (7.25) and (7.26) analytically (Braun 1993, 372), it is possible to show that solutions to the system exhibit certain qualitative properties. Also, we may assign values to the exogenous parameters in the system and obtain numerical solutions. In section 7.2 below, we consider the properties of the system (7.25) and (7.26). Following that, in the remaining sections of this chapter, we solve the system (7.25) and (7.26) numerically over a restricted domain, and examine in detail the qualitative and quantitative properties of such solutions.

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[229] We enclose "initial condition" in quotation marks here since such a condition customarily denotes the state of a *time-dependent* system at some initial time of interest t_0. Since it is the forward market price, p^f—rather than a time coordinate—that is our independent variable in this "timeless" problem, the notion of an initial time does not apply literally here. Nonetheless, we continue to refer to initial conditions in this problem.

On the existence and uniqueness of solutions, see subsection 7.2.2.

7.2 Properties of the system (7.25) and (7.26) and existence and uniqueness of solutions

7.2.1 Singularities

For this discussion, it is convenient to write the system (7.25) and (7.26) more compactly as follows. First, augment the vector of the duopolists' SFs (i.e., the dependent variables) with a third component[230] (only), defined as

$$\bar{S}_3^f\left(p^f\right) \equiv p^f, \tag{7.29}$$

which we may differentiate to yield

$$\bar{S}_3^{f\prime}\left(p^f\right) = 1. \tag{7.30}$$

Next, using eq. (7.29), define $\bar{S}^{f+}$ as an $(n+1)$x1 vector of the form[231]

$$\bar{S}^{f+}\left(p^f\right) \equiv \left(\bar{S}_1^f\left(p^f\right) \quad \bar{S}_2^f\left(p^f\right) \quad \bar{S}_3^f\left(p^f\right)\right)^\top,$$

which has the derivative with respect to p^f of

$$\bar{S}^{f+\prime}\left(p^f\right) = \left(\bar{S}_1^{f\prime}\left(p^f\right) \quad \bar{S}_2^{f\prime}\left(p^f\right) \quad 1\right)^\top. \tag{7.31}$$

We may then write eqs. (7.25), (7.26), and (7.30) in vector form as the system

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[230] Or, an $(n+1)^{\text{th}}$ component, for the general case of *n* firms.

[231] This augmentation of the vector of dependent variables permits us to suppress the explicit appearance of p^f in the ODE system; such systems are commonly called *autonomous* systems of ODEs. This step is helpful since many theoretical results for ODE systems are expressed with reference to such autonomous systems.

$$\mathscr{A}\left(\bar{S}^{f++}\left(p^f\right)\right)\bar{S}^{f+\prime}\left(p^f\right)=\mathscr{G}\left(\bar{S}^{f++}\left(p^f\right)\right), \tag{7.32}$$

where $\mathscr{A}\left(\bar{S}^{f++}\left(p^f\right)\right)$ is an $(n+1)\text{x}(n+1)$ matrix of the form

$$\mathscr{A}\left(\bar{S}^{f++}\left(p^f\right)\right)=\begin{bmatrix} \bar{S}^{f++}\left(p^f\right)^{\top}\mathscr{Q}_{12}\bar{S}^{f++}\left(p^f\right) & 0 & 0 \\ 0 & \bar{S}^{f++}\left(p^f\right)^{\top}\mathscr{Q}_{21}\bar{S}^{f++}\left(p^f\right) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{7.33}$$

and $\mathscr{G}\left(\bar{S}^{f++}\left(p^f\right)\right)$ is an $(n+1)\text{x}1$ vector of the form

$$\mathscr{G}\left(\bar{S}^{f++}\left(p^f\right)\right)=\begin{bmatrix} \bar{S}^{f++}\left(p^f\right)^{\top}\mathscr{Q}_{23}\bar{S}^{f++}\left(p^f\right) \\ \bar{S}^{f++}\left(p^f\right)^{\top}\mathscr{Q}_{13}\bar{S}^{f++}\left(p^f\right) \\ 1 \end{bmatrix}. \tag{7.34}$$

Since they contain quadratic forms, the matrix $\mathscr{A}\left(\bar{S}^{f++}\left(p^f\right)\right)$ and the vector $\mathscr{G}\left(\bar{S}^{f++}\left(p^f\right)\right)$ are each a quadratic function of the elements of $\bar{S}^{f++}\left(p^f\right)$.

Systems of the form of eq. (7.32) are often called *quasilinear* because in this case, we may write the general form of an implicit ODE, $\mathscr{F}\left(\bar{S}^{f++}\left(p^f\right),\bar{S}^{f+\prime}\left(p^f\right)\right)=0$, as

$$\mathscr{F}\left(\bar{S}^{f++}\left(p^f\right),\bar{S}^{f+\prime}\left(p^f\right)\right)\equiv\mathscr{A}\left(\bar{S}^{f++}\left(p^f\right)\right)\bar{S}^{f+\prime}\left(p^f\right)-\mathscr{G}\left(\bar{S}^{f++}\left(p^f\right)\right)=0, \tag{7.35}$$

where the (implicit) derivative term $\bar{S}^{f+\prime}\left(p^f\right)$ enters $\mathscr{F}\left(\bar{S}^{f++}\left(p^f\right),\bar{S}^{f+\prime}\left(p^f\right)\right)$ *linearly*. The system (7.32) is *singular* at a point $\left[\bar{S}_1^f\left(p^f\right),\bar{S}_2^f\left(p^f\right),p^f\right]$ when the matrix $\mathscr{A}\left(\bar{S}^{f++}\left(p^f\right)\right)$ in eq. (7.33) is not invertible, or singular. This occurs if and only if at least one of the quadratic forms on the diagonal of $\mathscr{Q}\left(\bar{S}^{f++}\left(p^f\right)\right)$ equals zero at that point, that is, when

$$\bar{S}^{f++}\left(p^f\right)^{\top}\mathscr{Q}_{12}\bar{S}^{f++}\left(p^f\right)=0 \tag{7.36}$$

or

$$\bar{S}^{f++}\left(p^f\right)^{\top}\mathscr{Q}_{21}\bar{S}^{f++}\left(p^f\right)=0\,, \tag{7.37}$$

or both. Recall from eq. (7.28), however, that the locus of points $\left[\bar{S}_1^f\left(p^f\right),\bar{S}_2^f\left(p^f\right),p^f\right]$ at which eq. (7.36) holds coincides exactly with the locus of such points at which eq. (7.37) holds. Thus we may consider exactly one of eqs. (7.36) and (7.37) to be redundant. We call points in this locus the *singular points*, or *singularities*, of the ODE system (7.32).[232] Pulling together this nomenclature, we may label eq. (7.32) a *singular quasilinear ODE system*.

Geometrically, the graphs of each equation (7.36) and (7.37) coincide in a common graph: a *quadratic surface*.[233] Because this quadratic surface is the locus of the singular points in this problem, we call this surface—defined by eqs. (7.36) and (7.37)—the *singular locus*. Informally, we may think of "most" points on the singular locus as that set of points at which, in the limit, both firms' forward market SFs become infinitely

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[232] To preserve symmetry in the discussion, however, we will customarily continue to refer to *both* of eqs. (7.36) and (7.37) as characterizing the system (7.32)'s singularities, although by the argument above, either equation (7.36) and (7.37), taken individually, would suffice to describe these points. In our earlier notation, a necessary and sufficient condition for eqs. (7.36) and (7.37) to hold is $\mathscr{P}_1^1\mathscr{P}_2^2-\mathscr{P}_2^1\mathscr{P}_1^2=0$, the *converse* of the restriction (7.22).

[233] One generates quadratic (or "quadric") surfaces by rotating a conic section about an axis of symmetry. On quadratic surfaces, see Eves (1987, 298) for a useful taxonomy, as well as Weisstein (1999a) and Hilbert and Cohn-Vossen (1952) for additional illustrations.

sloped.[234] Appendix E.2 examines in greater detail the theory and computation of singularities in the system (7.32).

7.2.2 *Solutions of the system (7.25) and (7.26) away from the singular locus*

At points not on the singular locus discussed in subsection 7.2.1,[235] we will have by definition that the converses of eqs. (7.36) and (7.37) will hold at all p^f, namely, (rewriting the restriction (7.27))

$$\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{12}\overline{S}^{f++}\left(p^f\right)\neq 0 \tag{7.38}$$

and (consistent with eq. (7.28)),

$$\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{21}\overline{S}^{f++}\left(p^f\right)\neq 0\,. \tag{7.39}$$

Under the conditions (7.38) and (7.39), $\mathcal{Q}\left(\overline{S}^{f++}\left(p^f\right)\right)$ is invertible, and we may write the system (7.32) in explicit form—that is, solving explicitly for the derivatives $\overline{S}_i^{f\prime}\left(p^f\right)$, $i=1,2$—as

$$\overline{S}_1^{f\prime}\left(p^f\right)=\frac{\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{23}\overline{S}^{f++}\left(p^f\right)}{\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{12}\overline{S}^{f++}\left(p^f\right)}, \tag{7.40}$$

$$\overline{S}_2^{f\prime}\left(p^f\right)=\frac{\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{13}\overline{S}^{f++}\left(p^f\right)}{\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{21}\overline{S}^{f++}\left(p^f\right)}, \tag{7.41}$$

---

[234] Assigning price p^f to the vertical axis, as usual, infinitely-sloped SFs would be parallel to the horizontal plane defined by the quantity axes $\overline{q}_1^f$ and $\overline{q}_2^f$. See Table 7.1 on page 252 below for a more precise discussion.

[235] We refer to such points as being "away from the singular locus."

and

$$\overline{S}_3^{f\prime}\left(p^f\right)=1\,. \tag{7.42}$$

If the inequalities (7.38) and (7.39) hold for every point on the SFs of interest, we obtain a *non-singular* ODE system (7.40)–(7.42). Given the aforementioned inequalities, SFs solving (7.40)–(7.42) do not intersect the singular locus (eqs. (7.36) and (7.37)) defined in section 7.2.1. Since the system (7.40)–(7.42) is non-singular, we may appeal to the standard theorems on existence, uniqueness and continuity of solutions to ODE systems (see, e.g., Birkhoff and Rota 1989, ch. 6 (in particular, Theorems 1, 2, 3, 8, 11, and applicable corollaries)). These theorems provide that, for the system (7.40)–(7.42), a unique solution exists—perhaps over a restricted domain of p^f—for any initial condition.[236] Moreover, such a solution is continuous, and varies continuously with the exogenous parameters of the problem.

The following section presents the computational methods used in this investigation to solve the system (7.40)–(7.42).

[236] Technically, the existence and uniqueness results apply to a *local solution* of the system (7.40)–(7.42) in the neighborhood of a given initial condition $\overline{S}^{f,0}\equiv\left[\overline{S}_1^f\left(p^{f,0}\right),\overline{S}_2^f\left(p^{f,0}\right),\,p^{f,0}\right]$. By "pasting together" such local solutions, we may extend such solutions to some *maximal interval of existence* $J_m\left(\overline{S}^{f,0}\right)\subseteq\mathbb{R}$, yielding a resulting *maximal* or *global solution* on $J_m\left(\overline{S}^{f,0}\right)$. See de la Fuente (2000, 437ff.) for details. We will not investigate the properties of solutions near the boundaries of intervals $J_m\left(\overline{S}^{f,0}\right)$, and so do not need to define them formally here.

7.3 Computational approaches to solving the differential equation system characterizing the forward market SFs

We used two distinct approaches to solving the differential equation system (7.40)–(7.42) characterizing the forward market SFs in the multi-settlement SFE model: (1) numerical integration using MATLAB (The MathWorks 2001), and (2) a difference equation approximation implemented here using Microsoft Excel's "Solver tool." These two approaches are complementary in that each highlights particular properties of solutions to the ODE system. This section provides details on both of these implementations of the multi-settlement SFE model.

7.3.1 Numerical integration using MATLAB

MATLAB offers several differential equation solvers for numerical solution of (non-singular) problems of the form (7.40)–(7.42), together with symbolic algebra capabilities (specifically, the Maple symbolic algebra kernel (Maplesoft 2002)).[237] We tested the performance of each of MATLAB's solvers on the present problem for reasonable ranges of parameters. The best-performing solver in terms of both stability and the range of prices over which we could integrate successfully is named "ode15s." Appendix E.2 discusses the properties of MATLAB's ode15s solver in greater detail.[238] This solver formed the core of the MATLAB-based solution to the system (7.40)–(7.42), to which we refer hereinafter as "the MATLAB model." Given an initial condition

[237] The MATLAB codes used in this thesis are available from the author.

[238] The solver ode15s worked best with the backwards differentiation formulae (BDFs) (rather than the numerical differentiation formulae (NDFs)) enabled. The BDFs are also commonly known as "Gear's method"; see Gear (1971). On the details of and the distinction between BDFs and NDFs, see Shampine and Reichelt (1997).

$\left[\bar{S}_1^f\left(p^{f,0}\right), \bar{S}_2^f\left(p^{f,0}\right), p^{f,0}\right]$—that is, initial quantities for each firm and a corresponding initial price[239]—we may use the MATLAB model to compute a trajectory $\bar{S}^f\left(p^f\right)$ in $\mathbb{R}^3$ that solves the system (7.40)–(7.42). Projecting this trajectory into the p^f-$\bar{q}_1^f$ and p^f-$\bar{q}_2^f$ planes, in turn, yields the SFs $\bar{S}_1^f\left(p^f\right)$ and $\bar{S}_2^f\left(p^f\right)$.[240]

7.3.2 *Difference equation approximation using the Excel Solver: The discrete Excel model*

The second computational approach that we employ in this investigation to solving the differential equation system (7.40)–(7.42) relies on a *difference equation approximation* to this system. Since this approach uses Microsoft Excel (Microsoft Corporation 2001)—in particular, Excel's Solver tool, hereinafter simply the "Excel Solver"[241]—we refer to this approach hereinafter as the "discrete Excel model."[242] In contrast to the MATLAB model's requirement of an exogenously-specified initial condition, we formulate the discrete Excel model to select endogenously a (locally) unique equilibrium trajectory, as elaborated below.

The discrete Excel model comprises a family of doubly-nested optimization problems having the general form

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[239] Recall from subsection 7.1.3 that we must specify an initial condition for an ODE to have a well-defined, unique solution. The MATLAB model requires that this initial condition be specified exogenously.

[240] As depicted, for example, in subsection 7.4.3's Figure 7.7 below.

[241] It appears that a similar discrete approximation of the system (7.40)–(7.42) could also have been implemented and solved in MATLAB by exploiting the capabilities of the "Optimization Toolbox," an add-on product for the MATLAB software suite. Because the discrete Excel model is relatively simple and effective, however, we did not attempt a MATLAB-based discretization of this problem.

[242] The Excel files used in this thesis are available from the author.

$$\min_{\substack{\bar{S}_i^f\left(p^f\right)\ (discretized) \\ \text{[Additional decision variables]}}} / \max \quad \text{Objective function}$$

$$s.t. \quad \text{Subgame-perfect Nash equilibrium in } \bar{\Sigma}_i^s \text{ and } \bar{S}_i^f \qquad (7.43)$$

$$s.t. \quad \text{Parameters } \Theta$$

$$\text{[Additional constraints]}$$

where, as discussed further below, the bracketed phrases in problem (7.43) indicate (optional) additional elements of the problem. We solve problem (7.43) using the Excel Solver.[243] In the following, we elaborate on the various components of this problem.

The elements of primary interest in problem (7.43) are the discretized values of $\bar{S}_i^f\left(p^f\right)$ ($i = 1, 2$) that represent the quantities offered by firm *i* over a specified vector of prices p^f. These price-quantity pairs constitute a piecewise affine spline approximation to a smooth forward market SF for each firm. The "[a]dditional decision variables" noted in problem (7.43) could be, for example, parameters of the problem for which market data and the literature offer little quantitative empirical support. Converting such parameters to decision variables in problem (7.43) would enable us to determine endogenous values for such parameters in this problem's solution.

We may solve problem (7.43) using a variety of objective functions. Two intuitively appealing choices for the objective function would be

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[243] According to Excel's documentation (in Excel, see "Help | About Solver"), "[t]he Microsoft Excel Solver tool uses the Generalized Reduced Gradient (GRG2) nonlinear optimization code developed by Leon Lasdon, University of Texas at Austin, and Allan Waren, Cleveland State University."

1. The minimization of the discrepancy between endogenous model outputs (e.g., expected prices and quantities in each market), and corresponding empirical reference values[244] from the literature
2. The maximization of expected aggregate welfare, which could be relevant as a benchmark for policy analysis

In addition, we could identify other plausible candidates for objective functions that correspond to special cases of the multi-settlement SFE model. For example, minimizing the "overall curvature" (defined in some meaningful way) of the forward market SFs might be used to identify forward market SFs that are (nearly) affine over a chosen price range. Choosing an objective function for problem (7.43) constitutes an *equilibrium selection rule* that identifies a single trajectory (assuming a unique solution for this problem) from the phase space of SF trajectories. Naturally, a different objective function would, in general, select a different SFE from this phase space.

The upper-level constraint set of problem (7.43) is itself a constrained equilibrium solution of the multi-settlement SFE model. The equilibrium constraint of "Subgame-perfect Nash equilibrium in $\bar{\Sigma}_i^s$ and $\bar{S}_i^f$" refers to the (simplified affine) spot and forward market equilibria described in chapters 4 and 5. This equilibrium comprises each firm's first- and second-order optimality conditions as well as slope restrictions on the forward market SFs. Here, the forward market equilibrium SFs are represented by the piecewise affine approximation corresponding to the discretized decision variables $\bar{S}_i^f\left(p^f\right)$ (i.e., quantities defined over a grid of fixed prices). We compute the subgame-perfect Nash

[244] For example, for California's electricity market during a particular period of interest. See Appendix F for details.

equilibrium subject to chosen parameter values—elements of the vector Θ—and possibly to "[a]dditional constraints."[245] Such additional constraints could include restrictions that enhance the verisimilitude of the model. Sections 7.5 and 7.6 below provide further details and specific examples of the application of problem (7.43).

7.3.3 Comparison of computational approaches

The discrete Excel model and the MATLAB model share some fundamental similarities. Like any numerical integration routine, the MATLAB model is at its heart also a discretization of what is—away from the singular locus—a continuously differentiable problem. The major algorithmic distinction between the two approaches lies in the discrete Excel model's incorporation of equilibrium selection—implemented using optimization problems having the general form of (7.43)—not represented in the MATLAB model. As a consequence, the Excel- and MATLAB-based approaches differ in their inputs and outputs in ways that are important for the present investigation. We review these distinctions below.

For our purposes, the discrete Excel model offers two distinct advantages over the MATLAB model described in subsection 7.3.1. First, by allowing for an equilibrium selection procedure, the discrete Excel model affords a systematic means of choosing the initial conditions $\left(\overline{q}_1^{f,0}, p^{f,0}\right)$ and $\left(\overline{q}_2^{f,0}, p^{f,0}\right)$ for each firm's forward market SF. Namely, the initial quantities $\overline{q}_1^{f,0} = \overline{S}_1^f\left(p^{f,0}\right)$ and $\overline{q}_2^{f,0} = \overline{S}_2^f\left(p^{f,0}\right)$ appear as simply

[245] To obtain numerical solutions, we assign values to the cost, demand, distributional, and risk parameters of the multi-settlement SFE model, already introduced in chapters 3 through 6. Together with some new notation, we collect these parameters as elements of a parameter vector Θ in subsection 7.4.1 below. Appendix F explains the provenance of the particular parameter values used to conduct the comparative statics and welfare analyses of this chapter.

two of the endogenously-determined decision variables in problem (7.43). The MATLAB model, in contrast, requires the user to specify exogenously the initial quantities for each firm. As a second advantage, it is straightforward in the discrete Excel model to impose explicitly the constraints that forward market SFs be strictly increasing, whereas this is not possible using only MATLAB's ODE solvers (see note 241, however). Finally, we note that the discrete Excel model permits the user to adjust both the (uniform) step size and—like MATLAB—the range of prices p^f considered.

The disadvantages of the discrete Excel model center around existence and uniqueness of solutions, and the ease with which we may solve the model to find solutions. First and most fundamentally, a feasible solution to the optimization problem (7.43) cannot always be found for a given set of constraints and decision variables. Trial and error[246] may be required to identify a model for which the Excel Solver can identify a feasible solution. If a feasible solution can be found, the Solver can guarantee only a locally optimal solution, not a globally optimal solution due to the nonlinearity of problem (7.43). Accordingly, the discrete Excel model's solution depends, in general, on the decision variables' initial values.[247] Finally, the MATLAB ODE solvers adjust dynamically the step size for numerical integration to keep the discretization error (see note 380 below) within acceptable limits, while the uniform step size in the discrete Excel model is fixed by the user. This implies that, at certain points, the approximation

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[246] That is, trial and error with respect to the following attributes: the constraint set, the set of decision variables, the grid of prices used in the approximation, initial values for the optimization, and parameters of the Excel solver (in Excel, see "Tools | Solver | Options | Help").

[247] Away from the singular locus, these questions of existence and uniqueness of solutions arise due to the nonlinear optimization problem in the discrete Excel model, rather than to theoretical properties of the ODE system (7.40)–(7.42). For (nonsingular) ODE systems, we recall that the theorems noted in subsection 7.2.2 guarantee the existence and uniqueness of solutions to such systems.

(discretization) error of the discrete Excel model can be relatively large.[248] Provided, however, that the discretized SFs in the chosen price range do not "straddle" any singularities (not necessarily the case in the trials that we examine), the approximation error may in principle be made arbitrarily small by increasing the number of price steps,[249] or decreasing the overall price range over which we compute the discretized SFs.

The MATLAB model (which exploits MATLAB's graphics capabilities) is well-suited to investigate qualitatively the phase space and the properties of SF trajectories starting from arbitrarily-specified initial conditions. Section 7.4 presents qualitative results from the MATLAB model; the analysis emphasizes the geometry of trajectories, the singular loci, and other salient features of the phase space.

7.4 Qualitative analysis of the differential equation system characterizing the forward market SFs

We begin in subsection 7.4.1 below by defining the general parameter vector Θ as well as a particular vector Θ^{base} whose elements serve as our set of base case parameter values for the multi-settlement SFE model. Subsection 7.4.2 then analyzes qualitatively the singular quasilinear ODE system, eq. (7.32). Following that, subsection 7.4.3 explores in greater detail the non-singular ODE system (7.40)–(7.42).

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[248] Two examples in which this approximation error tends to be large in magnitude are regions in which an SF's curvature is large, and at points on a segment of the affine approximation to an SF that are relatively distant from the endpoints of the segment (e.g., near the midpoint of such a segment). At the segment's endpoints, in contrast, the approximation is exact.

[249] The current implementation of the discrete Excel model characterizes a firm's forward market SF using eleven affine segments connecting twelve price-quantity pairs. Increasing the number of price steps is possible, in principle, though doing so would increase the size of the problem and hence its computation time.

7.4.1 The parameter vector Θ

We first introduce some new notation to represent the *elasticity of spot market demand*; denote this elasticity as e_{dem}^{s}. Given an empirical mean reference price $p_{empir}^{s,mean}$ and quantity $q_{empir}^{s,mean}$ for the spot market,[250] we may write e_{dem}^{s} in terms of $\gamma^{s} \equiv \partial D^{s}\left(p^{s}, \varepsilon^{s}\right) / \partial p^{s}$ [251] as

$$e_{dem}^{s} = -\frac{p_{empir}^{s,mean} \gamma^{s}}{q_{empir}^{s,mean}}. \tag{7.44}$$

We introduce the parameter e_{dem}^{s} in eq. (7.44) in order to conduct this chapter's quantitative analysis in terms of this intuitively more appealing parameter.

Let Θ be the (general) *parameter vector* for the multi-settlement SFE model, defined as

$$\Theta \equiv \begin{pmatrix} c_{01} & c_{02} & c_{1} & c_{2} & e_{dem}^{s} & \bar{\eta}_{R} & \sigma_{\eta_R}^{2} & \bar{\nu}_{R} & \sigma_{\nu_R}^{2} & \lambda_{R} \end{pmatrix}^{\top}. \tag{7.45}$$

The ten-element vector Θ collects the cost, distributional, and risk parameters already introduced in previous chapters, along with the demand elasticity parameter e_{dem}^{s} defined immediately above. Now denote as Θ^{base} the parameter vector Θ assuming *base case* values of each of its ten elements. The base case values of the cost function parameters c_{0i} and c_{i} ($i = 1, 2$) are based on empirical data from California's electricity market,

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[250] See the discussion of Appendix F.1.1 for data sources and values of $p_{empir}^{s,mean}$ and $q_{empir}^{s,mean}$.

[251] Recall that we wrote the forward market equilibrium optimality conditions (7.11) and (7.12) in subsection 7.1.1 above in terms of the slope parameter γ^{s} rather than the elasticity e_{dem}^{s}.

circa 1999, as detailed in Appendix F.1.3. The base case values of the elasticity e^{s}_{dem}, the four distributional parameters $\bar{\eta}_R$, $\sigma^2_{\eta_R}$, $\bar{\nu}_R$, and $\sigma^2_{\nu_R}$ (see section 6.5), and R's CARA parameter λ_R are endogenous to the benchmarking procedure for the discrete Excel model, described in subsection 7.5 below. Bringing together these exogenously- and endogenously-determined parameters in this problem, the resulting base case parameter vector Θ^{base} is,[252] to three significant figures,

$$\Theta^{base} \equiv \begin{pmatrix} c_{01} \\ c_{02} \\ c_1 \\ c_2 \\ e^{s}_{dem} \\ \bar{\eta}_R \\ \sigma^2_{\eta_R} \\ \bar{\nu}_R \\ \sigma^2_{\nu_R} \\ \lambda_R \end{pmatrix}^{base} = \begin{pmatrix} \$25.60/\text{MWh} \\ \$30.50/\text{MWh} \\ \$0.000341\big/(\text{MWh})^2 \\ \$0.00326\big/(\text{MWh})^2 \\ -5.95\text{e-}5 \\ 4640\ \text{MWh} \\ 2.46\text{e}6\ \text{MWh}^2 \\ 335\ \text{MWh} \\ 5.86\text{e}4\ \text{MWh}^2 \\ 3.20\text{e-}4\ \$^{-1} \end{pmatrix}. \tag{7.46}$$

Unless otherwise specified, the computations in this section rely on the base case parameter vector Θ^{base} of eq. (7.46).

7.4.2 The singular quasilinear ODE system, equation (7.32)

To study the singular quasilinear ODE system (7.32), it is useful to begin by characterizing two types of loci in this system's phase space. First, there is the singular locus defined by eqs. (7.36) and (7.37) and discussed in subsection 7.2.1 above. Roughly

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[252] See also eq. (F.39) and the associated discussion in Appendix F.

speaking, this is the locus at which (for "most" points in the locus—see Table 7.1 below) both firms' forward market SFs become, in the limit, infinitely sloped. For this reason, we also refer below to the singular locus as the ∞-*locus*. Second, we have the two loci at which, respectively, each of the first two elements of $\mathcal{G}\left(\overline{S}^{f++}\left(p^f\right)\right)$ vanishes (see eq. (7.34)), that is, the locus represented by the equation

$$\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{23}\overline{S}^{f++}\left(p^f\right)=0 \tag{7.47}$$

and that corresponding to the equation

$$\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{13}\overline{S}^{f++}\left(p^f\right)=0\,. \tag{7.48}$$

For convenience, we refer to the loci (7.47) and (7.48) as the 0_1-*locus* ("zero-one locus") and the 0_2-*locus* ("zero-two locus"), since at non-singular points in these loci (again, see Table 7.1 below), we have that $\overline{S}_1^{f\prime}\left(p^f\right)=0$ and $\overline{S}_2^{f\prime}\left(p^f\right)=0$, respectively. For ease of reference, we collect this terminology in Table 7.1 below.

TABLE 7.1: LOCI OF INTEREST IN THE SINGULAR QUASILINEAR ODE SYSTEM (7.32)

Name of the locus	*Equation(s) characterizing the locus*	*Properties satisfied by "most" points[a] on the locus*
∞-locus (also, "singular locus")	$\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{12}\bar{S}^{f++}\left(p^f\right)=0$ or $\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{21}\bar{S}^{f++}\left(p^f\right)=0$ [253]	$\bar{S}_i^{f\prime}\left(p^f\right)\to\infty$, $i=1,2$
0_1-locus	$\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{23}\bar{S}^{f++}\left(p^f\right)=0$	$\bar{S}_1^{f\prime}\left(p^f\right)=0$
0_2-locus	$\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{13}\bar{S}^{f++}\left(p^f\right)=0$	$\bar{S}_2^{f\prime}\left(p^f\right)=0$

Note:

[a] In the restriction to "most" points, we exclude those points lying on the manifolds at which either (1) the ∞- and 0_1-loci, or (2) the ∞- and 0_2-loci intersect. We would need to determine the slopes $\bar{S}_i^{f\prime}\left(p^f\right)$ at such points on a case-by-case basis; the generalizations in the rightmost column of the table do not necessarily apply. On the other hand, for points at the manifold of intersection of the 0_1- and 0_2-loci (but not also on the ∞-locus), we have that $\bar{S}_1^{f\prime}\left(p^f\right)=0$ and $\bar{S}_2^{f\prime}\left(p^f\right)=0$ (as the table indicates). We include the generalizations in the rightmost column of the table solely as an aid to intuition, and emphasize that, without exception, we characterize the loci using the equations in the middle column of the table.

Where appropriate in the discussion below, we refer generically to the 0_1-locus or the 0_2-locus as a 0_i*-locus* ("zero-eye locus").

This subsection characterizes each of Table 7.1's loci analytically using the taxonomy of Eves (1987, 298) for quadratic forms, and plots their graphs using MATLAB's three-dimensional visualization capabilities (and assuming, unless otherwise specified, that $\Theta=\Theta^{base}$). Eves' taxonomy associates relationships among a quadratic form's coefficients—for example, the elements of $\mathcal{Q}_{ij}$ in each of Table 7.1's quadratic forms—with one of the seventeen types of quadratic surfaces. The taxonomy involves

[253] Recall that these two equations are redundant; hence, we use the conjunction "or."

rank, determinantal, and eigenvalue conditions of the coefficient matrices associated with each quadratic form.

Consider first the ∞-locus. Applying the taxonomy of Eves (1987, 298) to (either) equation representing this locus, we may show that this locus is a *real elliptic (double) cone*. Figure 7.1 below depicts the ∞-locus, confirming this classification.

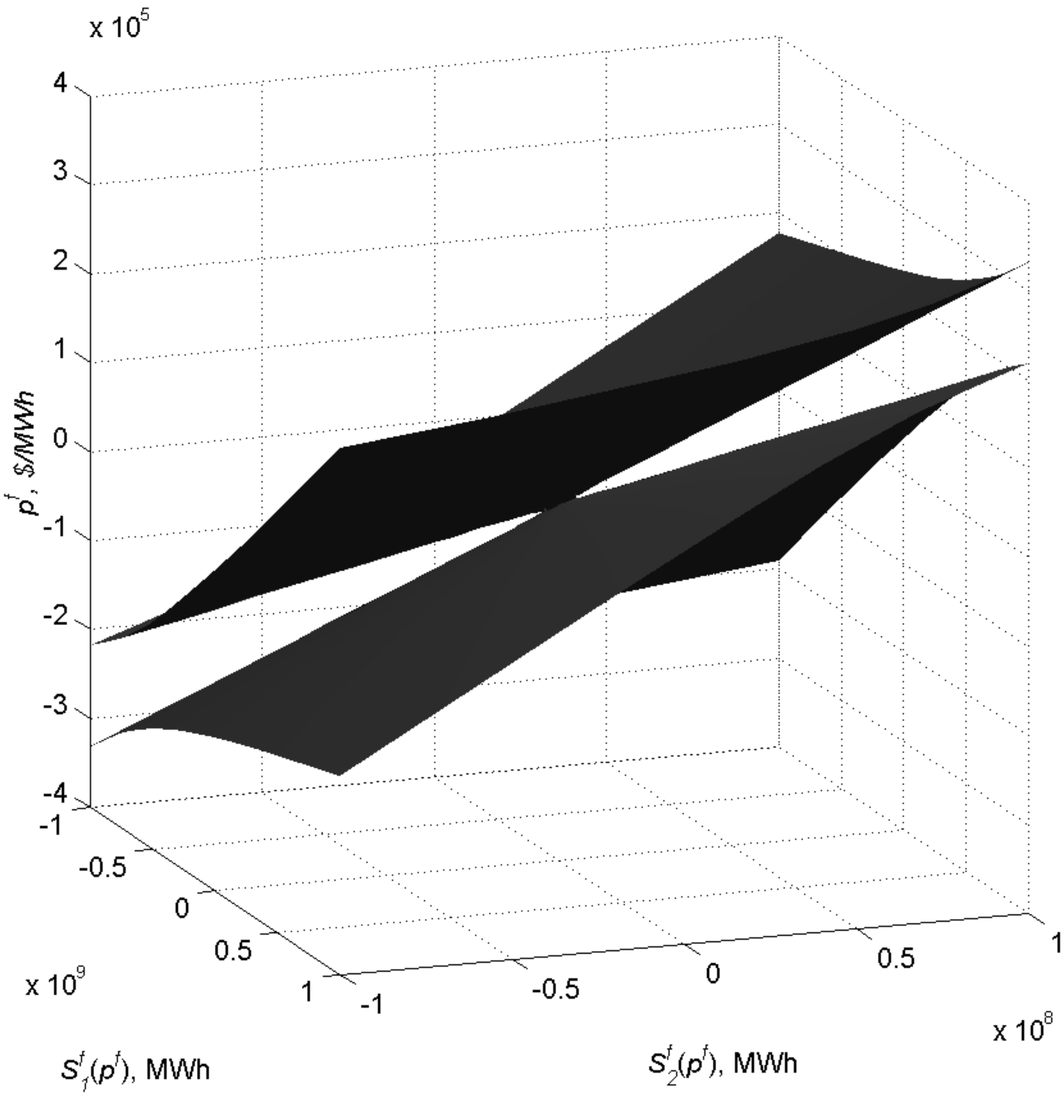

FIGURE 7.1: THE ∞-LOCUS, A REAL ELLIPTIC (DOUBLE) CONE, IN A NEIGHBORHOOD OF THE ORIGIN

The graph depicted in Figure 7.1 is, naturally, only a discrete approximation—made for the sake of visualization—to a theoretical real elliptic (double) cone. The fact that the two nappes of the cone do not appear to meet at a single point—the vertex—but rather appear to intersect over a continuum of points is merely an artifact of this discretization.[254]

[254] Refining the resolution of the lattice used to visualize the cone "shrinks" the apparent continuum at which the nappes of the cone meet. This behavior is consistent with the familiar theoretical property that the cone's two nappes meet at a point.

Next, we examine the 0_1-locus. The taxonomy of Eves (1987, 298) implies from the corresponding equation that this locus is a *hyperboloid of one sheet*. Figure 7.2 below depicts the 0_1-locus, confirming this result.

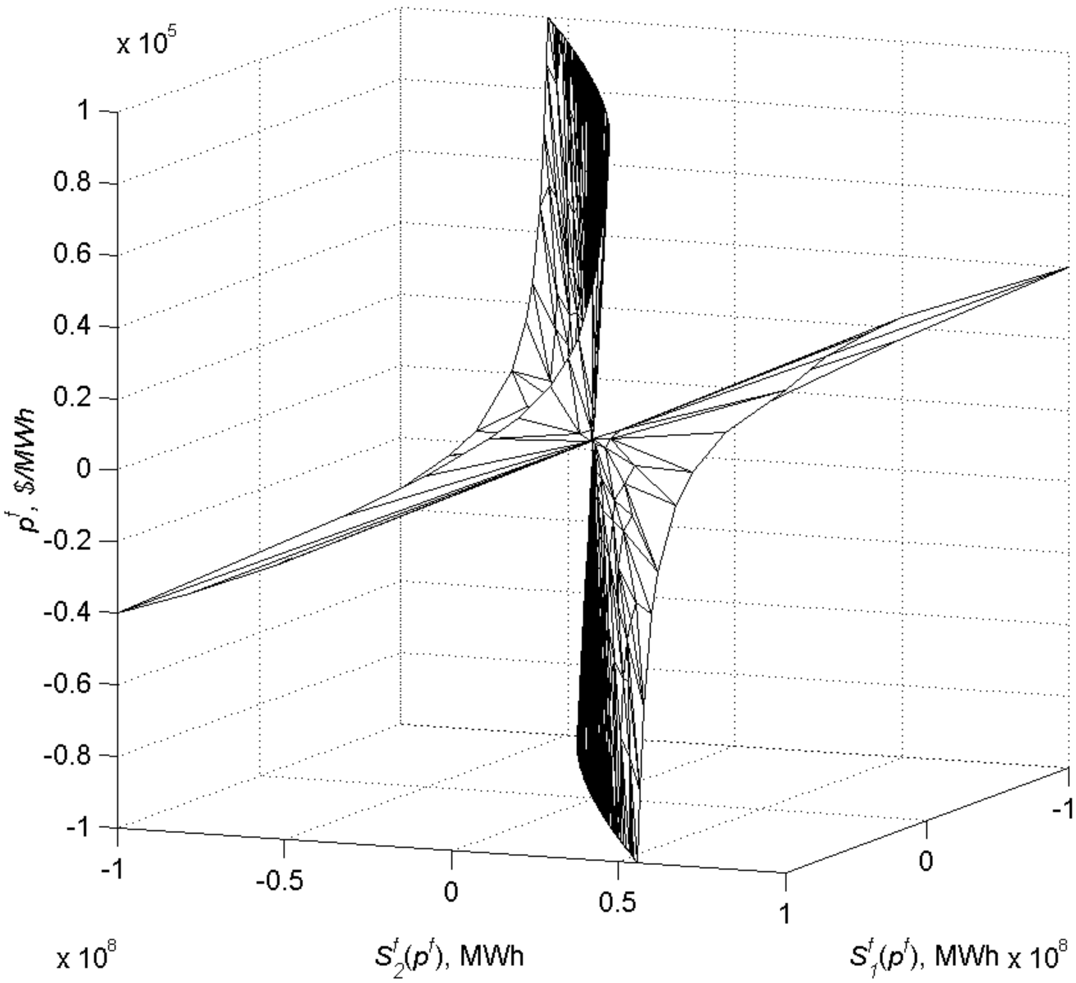

FIGURE 7.2: THE 0_1-LOCUS, A HYPERBOLOID OF ONE SHEET, IN A NEIGHBORHOOD OF THE ORIGIN

Finally, applying the taxonomy of Eves (1987, 298) to the equation representing the 0_2-locus, we find that this locus is also a *hyperboloid of one sheet*. Figure 7.3 below depicts the 0_2-locus, again confirming this result.

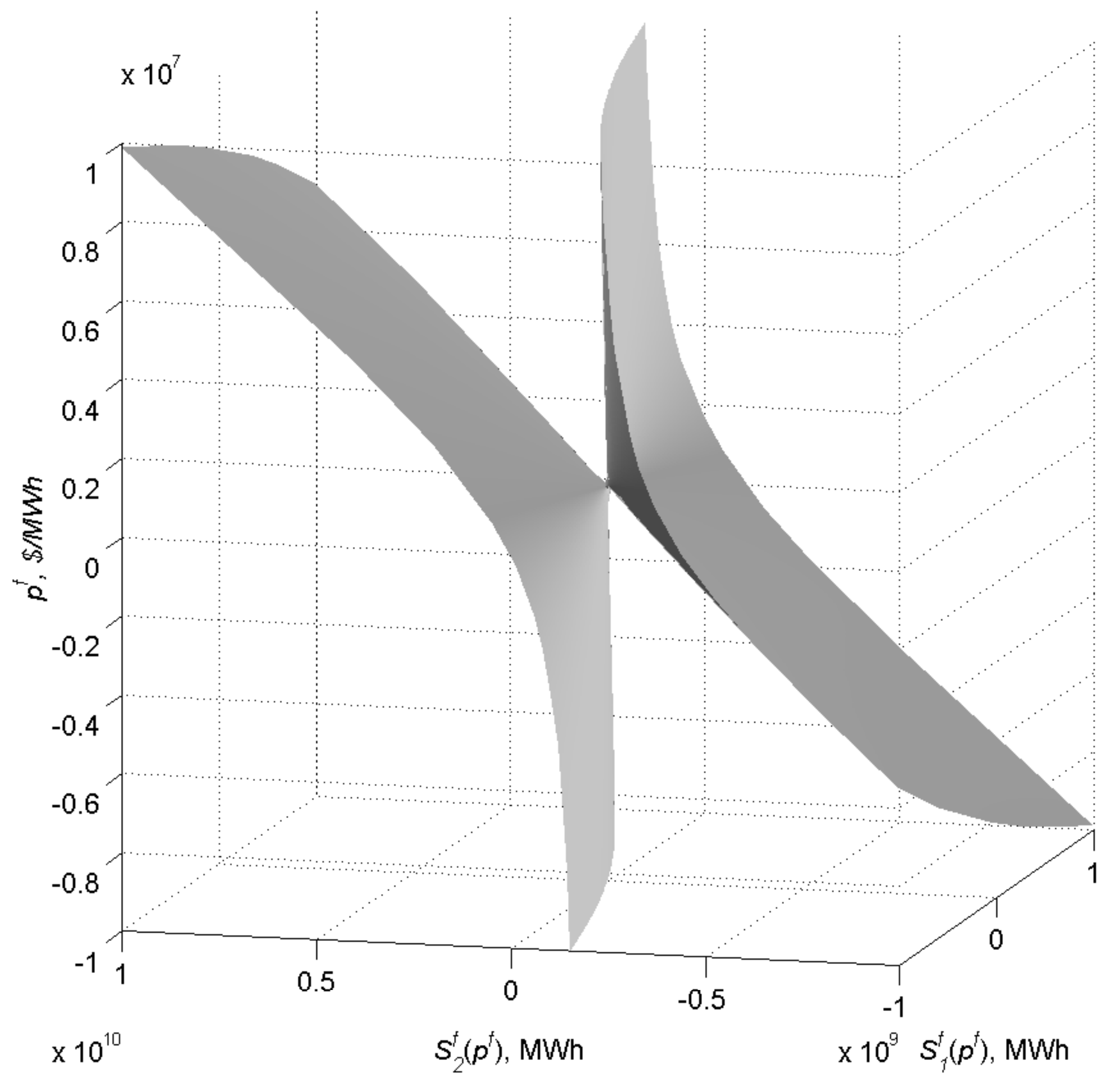

FIGURE 7.3: THE 0_2-LOCUS, A HYPERBOLOID OF ONE SHEET, IN A NEIGHBORHOOD OF THE ORIGIN

To emphasize the geometry of the three loci in Figure 7.1–Figure 7.3, we drew these figures to a smaller scale than would be appropriate to depict equilibria in the California electricity market (see Appendix F.2 for representative forward market quantities). Next, in Figure 7.4 below, we superimpose the graphs of the loci shown in Figure 7.1–Figure 7.3, enlarging the scale of the plot, as well.

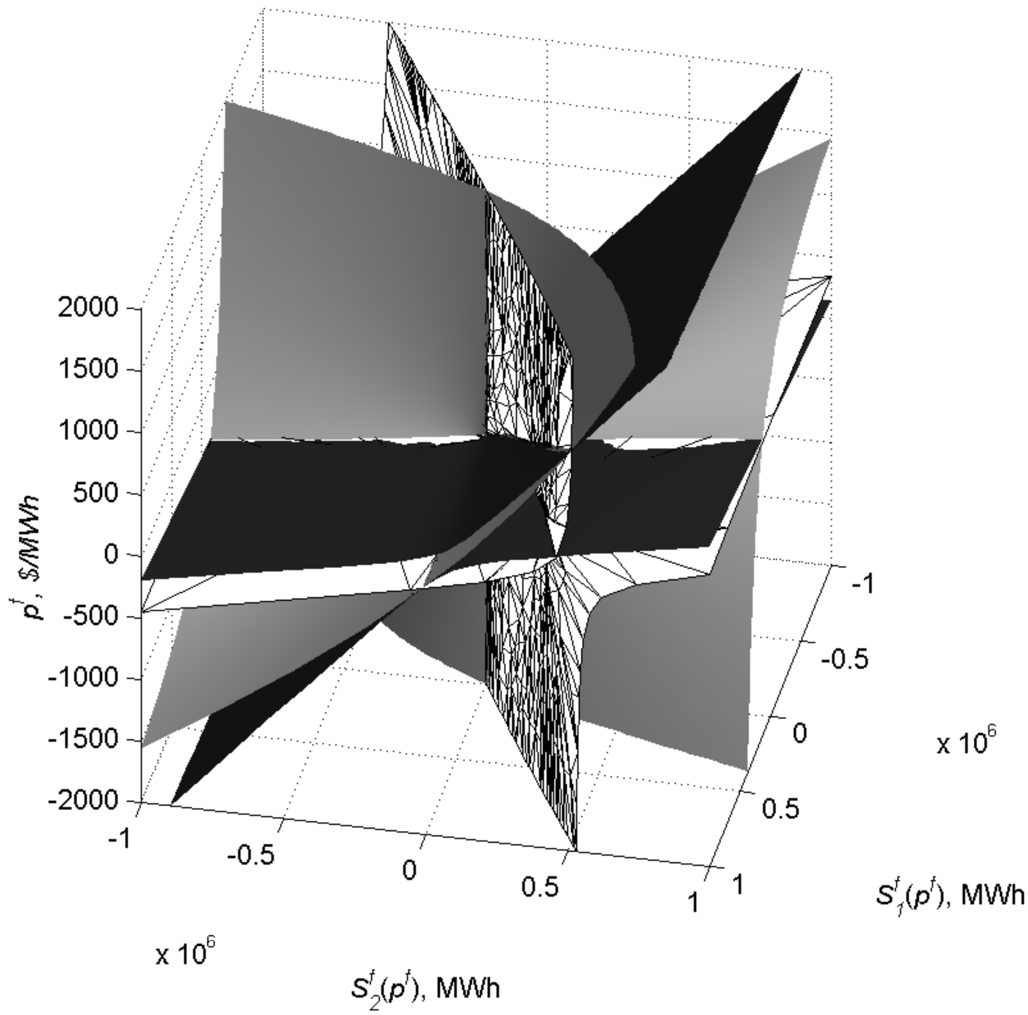

FIGURE 7.4: THE ∞-LOCUS (IN BLACK), THE 0_1-LOCUS (A TRIANGULAR MESH), AND THE 0_2-LOCUS (IN GRAY) IN A (SMALLER) NEIGHBORHOOD OF THE ORIGIN

In the next few figures, for clarity, we suppress both 0_i-loci and examine the relationships between various SF trajectories and the ∞-locus. Recall, as Figure 7.1 depicts, that the ∞-locus—the black surface in Figure 7.4—is a real elliptic (double) cone. Given that the orientation of the cone's axis (a function of the parameters Θ) is more nearly parallel with the vertical (p^f) axis than with either quantity axis, it is natural to characterize this ∞-locus as dividing the phase space into three *partitions*: the *upper partition*, the *middle partition*, and the *lower partition*. Figure 7.5 below portrays the ∞-locus along with separate SF trajectories beginning in each of these three partitions.[255]

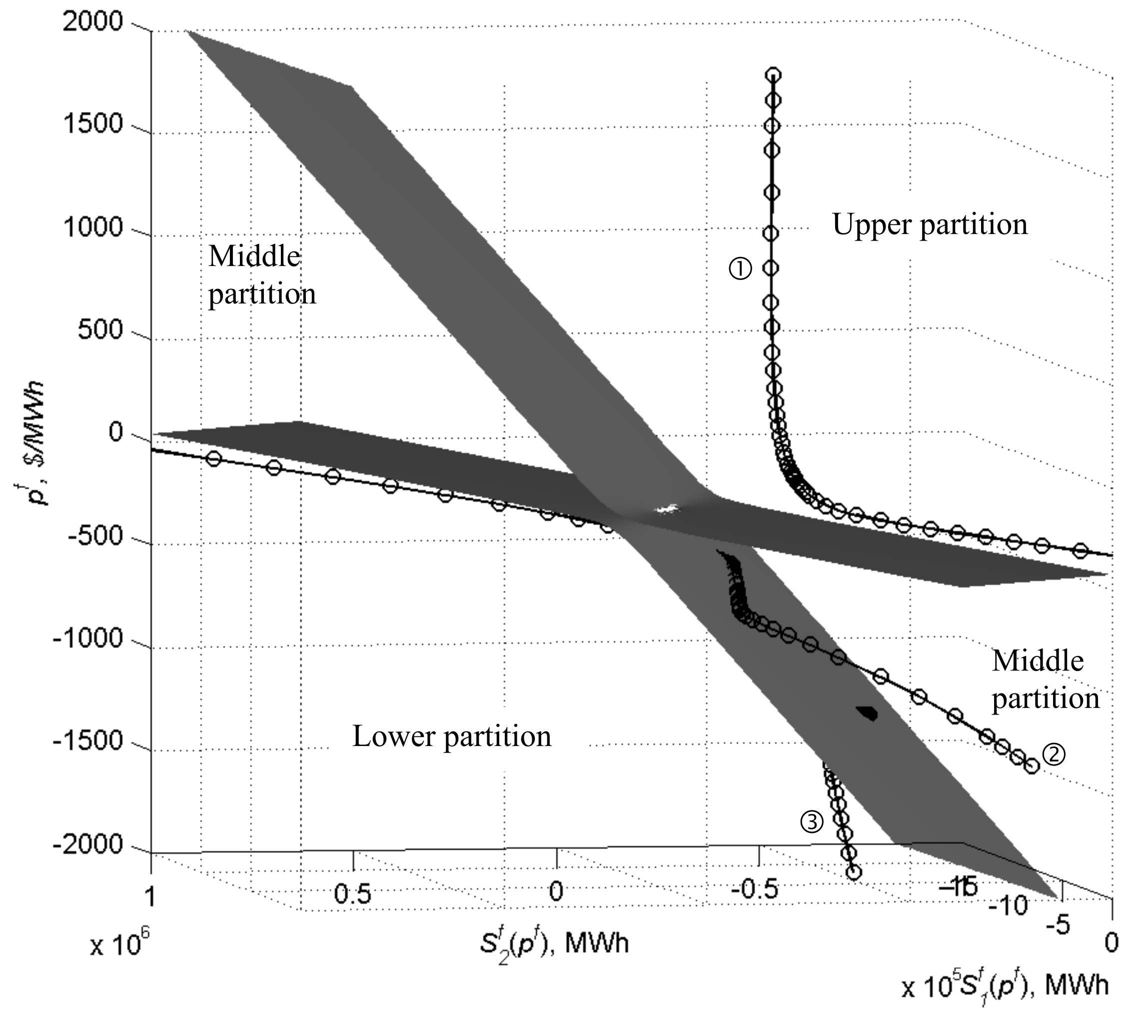

FIGURE 7.5: THE ∞-LOCUS (BLACK SURFACE) DIVIDING THE PHASE SPACE INTO UPPER, MIDDLE, AND LOWER PARTITIONS, AND SF TRAJECTORIES (BLACK CURVES MARKED WITH "O") BEGINNING IN EACH PARTITION

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[255] We define the partitions of the phase space as open sets, bounded, in part, (as depicted in Figure 7.5) by the ∞-locus (and otherwise unbounded). By this definition, points on the ∞-locus itself belong to none of these partitions, and therefore each partition contains exclusively *non-singular* points.

Figure 7.5 illustrates the three types of behavior that we have observed for forward market SF trajectories as they approach the ∞-locus.[256] Namely, an SF trajectory can be

- deflected by,
- transverse to, or
- absorbed by

the ∞-locus.[257] Figure 7.5 depicts three distinct trajectories in the neighborhood of the ∞-locus, each of which begins in a different phase space partition and each of which exhibits one of the three behaviors noted above.[258] We characterize these behaviors informally below.

Consider first the trajectory depicted in the upper partition of Figure 7.5, labeled as "①." Qualitatively, we may say that the ∞-locus *deflects* this trajectory, that is, the direction of this trajectory changes abruptly in the vicinity of the ∞-locus.

Next, consider the trajectory labeled as "②" in Figure 7.5, which begins in the middle partition at $p^f = -\$1{,}500/\text{MWh}$ and moves up (i.e., in the direction of increasing p^f) and to the left from there. This trajectory crosses the ∞-locus—we say it is

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[256] The trajectories depicted in Figure 7.5 do not necessarily satisfy the second-order condition for optimality for either firm over the entire price range; we use these trajectories for expository purposes only. This is the case (unless otherwise specified) for SFs and trajectories portrayed in all of this subsection's figures.

[257] While we do not claim that the above classification of behavior is exhaustive of all possibilities, all trajectories investigated in this study clearly fell into one of these three categories, as defined below.

[258] The apparent correspondence in Figure 7.5 between the partitions and the three trajectory behaviors discussed here is incidental. For different initial conditions or parameter values, we can find trajectories in each partition that behave differently.

transverse to the ∞-locus—and continues into the lower partition. Closer numerical examination of trajectory ② reveals that the slopes $\bar{S}_i^{f\prime}\left(p^f\right)$ near the apparent intersection with the ∞-locus are on the order of 10^3, that is, these slopes are clearly finite. This observation suggests that at the ∞-locus, trajectory ② encounters a *removable singularity*,[259] meaning that the magnitudes of the SF slopes $\bar{S}_1^{f\prime}\left(p^f\right)$ and $\bar{S}_2^{f\prime}\left(p^f\right)$ in eqs. (7.40) and (7.41) are bounded along an SF trajectory in the neighborhood of the singularity. Consequently, the MATLAB ODE solver does not fail in this neighborhood, making numerical integration using our model feasible almost everywhere—that is, on "both sides" of the ∞-locus.[260] This finding is supported by further graphical investigation (not illustrated in Figure 7.5), which indicates that the intersection of this trajectory with the ∞-locus is also close to a point at which the ∞-, 0_1-, and 0_2-loci all appear to intersect. While we would require further research to

[259] A removable singularity of a real function $f\left(x\right)$ is a singular point x_0 at which we may assign a value $f\left(x_0\right)$ such that *f* is *analytic*, that is, *f* possesses derivatives of all orders and agrees with its Taylor series in the neighborhood of every point (Weisstein 1999b, 1999c).

[260] A more familiar example of a removable singularity is found in Green (1999a). In Green's Figure 1 (p. 114), removable singularities exist at the points of intersection of the marginal cost function and his spot market supply functions, that is, at quantities X1 and X2 for the supply functions S(X1) and S(X2), respectively. To see this analytically, solve Green's eq. (4) for dq_j / dp to obtain

$$\frac{dq_j}{dp} = \frac{q_i\left(p\right) - x_i}{p - c_i q_i\left(p\right)} - b \, .$$

The ratio $\left(q_i\left(p\right) - x_i\right) / \left(p - c_i q_i\left(p\right)\right)$ in the above equation is *indeterminate* at the quantities X1 and X2 noted above (i.e., at the removable singularities), but it may be evaluated via L'Hopital's Rule. Similarly, a removable singularity exists at the origin in Klemperer and Meyer's (1989) connected set of SFEs; see their Figure 1 (reproduced as Figure 7.10 below) and their eq. (5).

corroborate numerically that this represents an actual point of intersection,[261] these observations suggest that the singular point at which trajectory ② crosses the ∞-locus belongs to a class of more complex singularities. These more complex singularities likely differ in important ways from other points on the ∞-locus, exemplified by the possibility of trajectories transverse to the ∞-locus at such points.

Finally, regarding the trajectory labeled as "③" in the lower partition of Figure 7.5, we may say that the ∞-locus *absorbs* this trajectory. More precisely, in this case, the MATLAB solver fails and numerical integration halts (see Appendix E.3 for details) when the trajectory approaches the ∞-locus sufficiently closely. This numerical failure is due, analytically, to the derivatives $\bar{S}_i^{f\prime}\left(p^f\right) \to \infty$ that explode as trajectory ③ approaches the ∞-locus (recall Table 7.1 above).

While a closer analysis of the factors governing trajectories' behavior in the neighborhood of the ∞-locus is left for further research,[262] we make here a few general observations on these factors. In theoretical terms, the vector field corresponding to an underlying ODE system is tangent to any arbitrary solution trajectory at all points along the trajectory. Accordingly, the parameter values that determine this vector field will clearly contribute to determining how trajectories behave in different regions of the phase

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[261] Or, practically speaking, a small neighborhood through which the trajectory and the various surfaces pass, since we are dealing invariably with approximate numerical representations of the underlying theoretical objects.

[262] Riaza (2002, 306) highlights the distinction made in the applied mathematics literature between *algebraic singularities* (where, in our framework, $\mathcal{G}\left(\bar{S}^{f++}\left(p^f\right)\right) \notin \operatorname{rge} \mathcal{Q}\left(\bar{S}^{f++}\left(p^f\right)\right)$, recalling eqs. (7.33) and (7.34)), and *geometric singularities* (where $\mathcal{G}\left(\bar{S}^{f++}\left(p^f\right)\right) \in \operatorname{rge} \mathcal{Q}\left(\bar{S}^{f++}\left(p^f\right)\right)$). Exploring this distinction in the present context may be a useful point of departure for future work. See also the discussion of Appendix E.2, which examines in greater detail the theory and computation of solution trajectories in the neighborhood of this model's singularities.

space. Each trajectory that we compute in this chapter is, naturally, a numerical approximation to an underlying "theoretical" trajectory. Error tolerances and step size restrictions for numerical integration will play a role in determining the extent, the accuracy of approximation near the singularities, and perhaps even which of the three behaviors identified above that a (numerical) trajectory exhibits. In some cases, the numerical approximation may only approximate the theoretical trajectory over a limited range. For example, while Figure 7.5 showed that trajectory ③ was apparently absorbed by the ∞-locus, simply stopping short before reaching this locus, this behavior is clearly attributable—as MATLAB error messages report—to a numerical rather than a theoretical cause (i.e., failure of the MATLAB solver). Therefore, although the numerical trajectory stops near the ∞-locus, it is certainly possible that the underlying theoretical trajectory extends beyond this point. Through a change of coordinates to remove the singularity, it may also be possible to extend such a trajectory numerically, through the ∞-locus.[263] Again, we reserve for future research the exploration of such questions.

A special case is the situation in which suppliers' cost functions and initial conditions are *symmetric*. As a specific illustration, define a symmetric parameter vector Θ^{symm} as the vector Θ^{base} (recall eq. (7.46)) with firm 2 replaced by a replica of firm 1 in the base case, so that $\left(c_{02}\right)^{symm} = \left(c_{01}\right)^{base}$ and $\left(c_2\right)^{symm} = \left(c_1\right)^{base}$. That is, we define Θ^{symm} as

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[263] For example, interchanging the dependent and independent variables would imply that SF slopes would now approach zero at points where they were formerly unbounded. Accordingly, the ODE solver would no longer fail at such points. The author is indebted to Allan Wittkopf of Maplesoft (Wittkopf 2002) for suggesting this approach to numerical integration in the vicinity of such singularities, and for helpful discussions concerning computational implementation using MAPLE (Maplesoft 2002).

$$\Theta^{symm} \equiv \begin{pmatrix} c_{01} \\ c_{02} \\ c_1 \\ c_2 \\ e^s_{dem} \\ \bar{\eta}_R \\ \sigma^2_{\eta_R} \\ \bar{\nu}_R \\ \sigma^2_{\nu_R} \\ \lambda_R \end{pmatrix}^{symm} = \begin{pmatrix} \$25.60/\text{MWh} \\ \$25.60/\text{MWh} \\ \$0.000341/(\text{MWh})^2 \\ \$0.000341/(\text{MWh})^2 \\ -5.95\text{e-}5 \\ 4640 \text{ MWh} \\ 2.46\text{e}6 \text{ MWh}^2 \\ 335 \text{ MWh} \\ 5.86\text{e}4 \text{ MWh}^2 \\ 3.20\text{e-}4 \text{ \$}^{-1} \end{pmatrix}. \tag{7.49}$$

Also take the initial condition to be

$$\left[\bar{S}_1^f\left(p^{f,0}\right), \bar{S}_2^f\left(p^{f,0}\right), p^{f,0}\right] = \left[1\text{e}6 \text{ MWh}, 1\text{e}6 \text{ MWh}, \$2000/\text{MWh}\right],$$

symmetric across the two firms.

Figure 7.6 below illustrates the resulting trajectory, integrating downward over the range $p^f \in [\$2000/\text{MWh}, -\$2000/\text{MWh}]$.

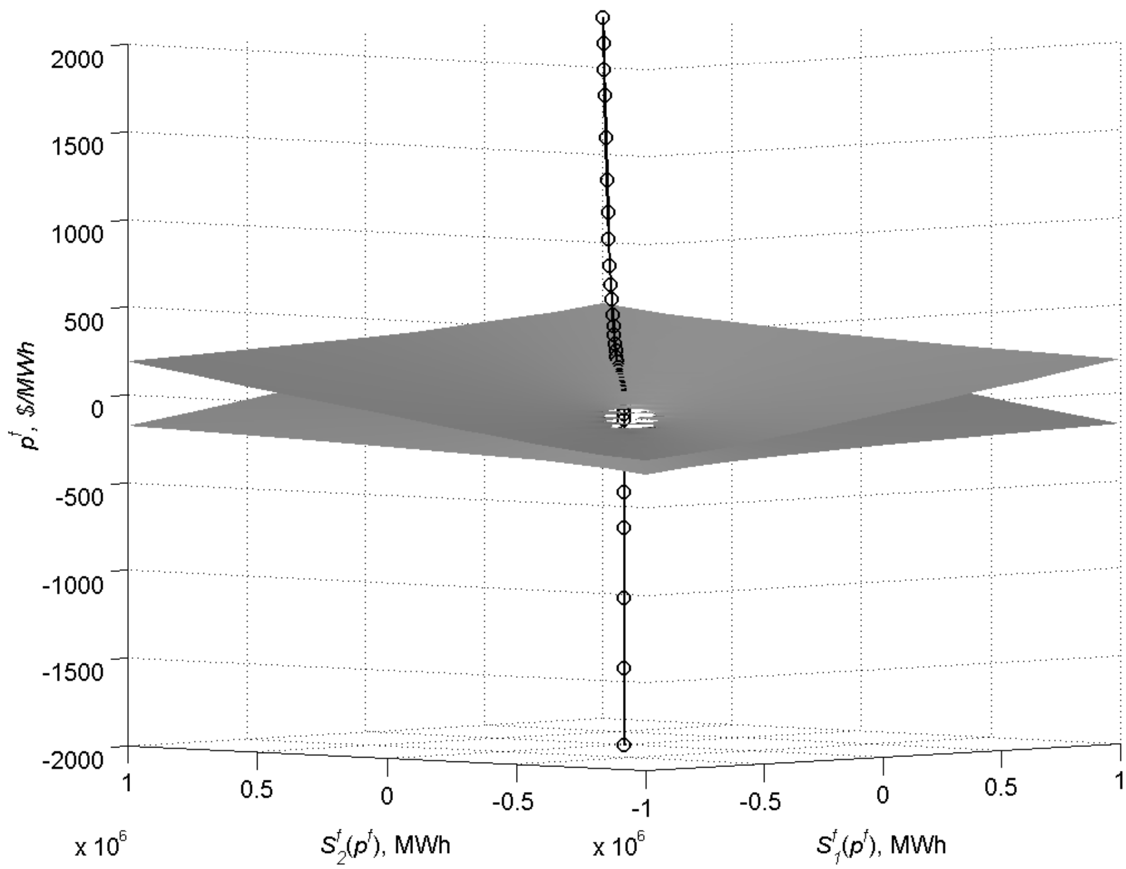

FIGURE 7.6: WITH SYMMETRIC SUPPLIERS (THAT IS, ASSUMING $\Theta = \Theta^{symm}$ FROM EQ. (7.49)) AND SYMMETRIC INITIAL CONDITIONS, THE SF TRAJECTORY IS TRANSVERSE TO THE ∞-LOCUS

In this symmetric case, we observe that, like trajectory ② of Figure 7.5, firms' trajectories are also transverse to the ∞-locus. The SF trajectory depicted in Figure 7.6 appears to cross the ∞-locus near the vertex of the double cone.[264] In analytical terms,

[264] Whether this is indeed a *general* property of trajectories in the symmetric case is a question reserved for future research.

we conjecture that the numerators and denominators of eqs. (7.40) and (7.41)[265] will go to zero at same rate along an SF trajectory as it approaches the ∞-locus. This property implies that under symmetry, all of the singularities of the ∞-locus become removable singularities (see note 259).

Return again to the base case parameters Θ^{base} and Figure 7.5 above, which portrays various trajectories $\overline{S}^{f}\left(p^{f}\right)$ from different initial conditions and ranges of integration. In accordance with the a priori slope constraints (due to market rules defining admissible SFs) noted in subsection 3.1.5, we are interested in locating and characterizing a forward market SF for each firm that is *strictly increasing*. To simplify the search for strictly increasing SFs and for ease of exposition, we restrict the qualitative analysis for the remainder of this section (and the numerical analysis in the rest of this chapter) to SFs inhabiting the *upper partition* of the phase space (see Figure 7.5 above). In the analysis below, we are able to identify SFs in the upper partition that slope upward, at least over certain price ranges. We may implement the restriction to consider only SFs inhabiting the upper phase space partition through judicious choice of the SFs' initial conditions. In particular, we may exploit the observed empirical regularity that an SF trajectory beginning at an initial condition $\overline{S}^{f,0} \equiv \left[\overline{S}_1^{f}\left(p^{f,0}\right), \overline{S}_2^{f}\left(p^{f,0}\right), p^{f,0}\right]$ within the upper phase space partition remains in the upper partition for any chosen range of integration.[266]

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[265] Under symmetry, these two equations are, of course, identical.

[266] Although this observation remains unproven as a theoretical matter, we observed this behavior in numerical trials, without exception. Naturally, if the specified range of integration includes prices outside of the maximal interval of existence $J_m\left(\overline{S}^{f,0}\right)$ (see note 236), the solver will fail to find a solution

Restricting our attention to the upper partition of the phase space is a substantive limitation in the scope of this analysis, though extending it to include SFs in the other partitions would be straightforward. That is, we could in principle undertake an analysis of trajectories inhabiting the middle and lower partitions of the phase space similar to that conducted below for trajectories in the upper partition. Indeed, preliminary explorations confirm that, as in the upper partition, there are regions within the middle and lower partitions in which both firms' SFs slope upward. Moreover, it may be reasonable to suppose that SF trajectories lying in these other partitions share the characteristics of those trajectories in the upper partition that we study here (e.g., the comparative statics properties discussed in section 7.6 below). We reserve for future research, however, such questions pertaining to SF trajectories lying in the middle and lower partitions of the phase space, and do not consider further these trajectories in the present work.

Accordingly, the next subsection below restricts the analysis to trajectories inhabiting the upper partition which, by construction (recall note 255), contains only non-singular points. In accordance with this restriction, we supplant the singular quasilinear ODE system (eq. (7.32)) with the non-singular ODE system (eqs. (7.40)–(7.42)) as the object of our analysis.

7.4.3 *The upper partition of the phase space of the non-singular ODE system, equations (7.40)–(7.42)*

In this subsection, we focus on the portions of the 0_i-loci and trajectories that lie within the upper partition of the phase space depicted in Figure 7.5. As argued at the close of

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at such prices. Moreover, the solver may fail to compute a numerical solution as the theoretical trajectory approaches the ∞-locus sufficiently closely from within the upper partition.

the previous subsection, confining our attention to the upper partition permits us to replace eq. (7.32) with the non-singular ODE system, equations (7.40)–(7.42). We also further enlarge the scales of the plots below to consider forward market quantities for each firm in the range $[-1e4, 1e4]$ MWh. This range is representative of actual forward market quantities observed in the California electricity market. Using the system (7.40)–(7.42), we study the upper partition of the phase space to identify admissible—in particular, strictly increasing—SFs.

As subsection 7.3.1 noted, for each firm *i*, the forward market SF $\bar{S}_i^f\left(p^f\right)$ that solves the system (7.40)–(7.42) is simply the projection of the trajectory $\bar{S}^f\left(p^f\right)$ in $\mathbb{R}^3$ into the p^f-q_i^f plane. Figure 7.7 below shows (as dashed lines) these planar projections for firms 1 and 2 of an SF trajectory $\bar{S}^f\left(p^f\right)$ (the solid line) lying in the upper partition (that is, above the ∞-locus, the black surface in the figure). In the figure, we see that the ∞-locus deflects this particular SF trajectory in the neighborhood of $p^f = \$40/\text{MWh}$.

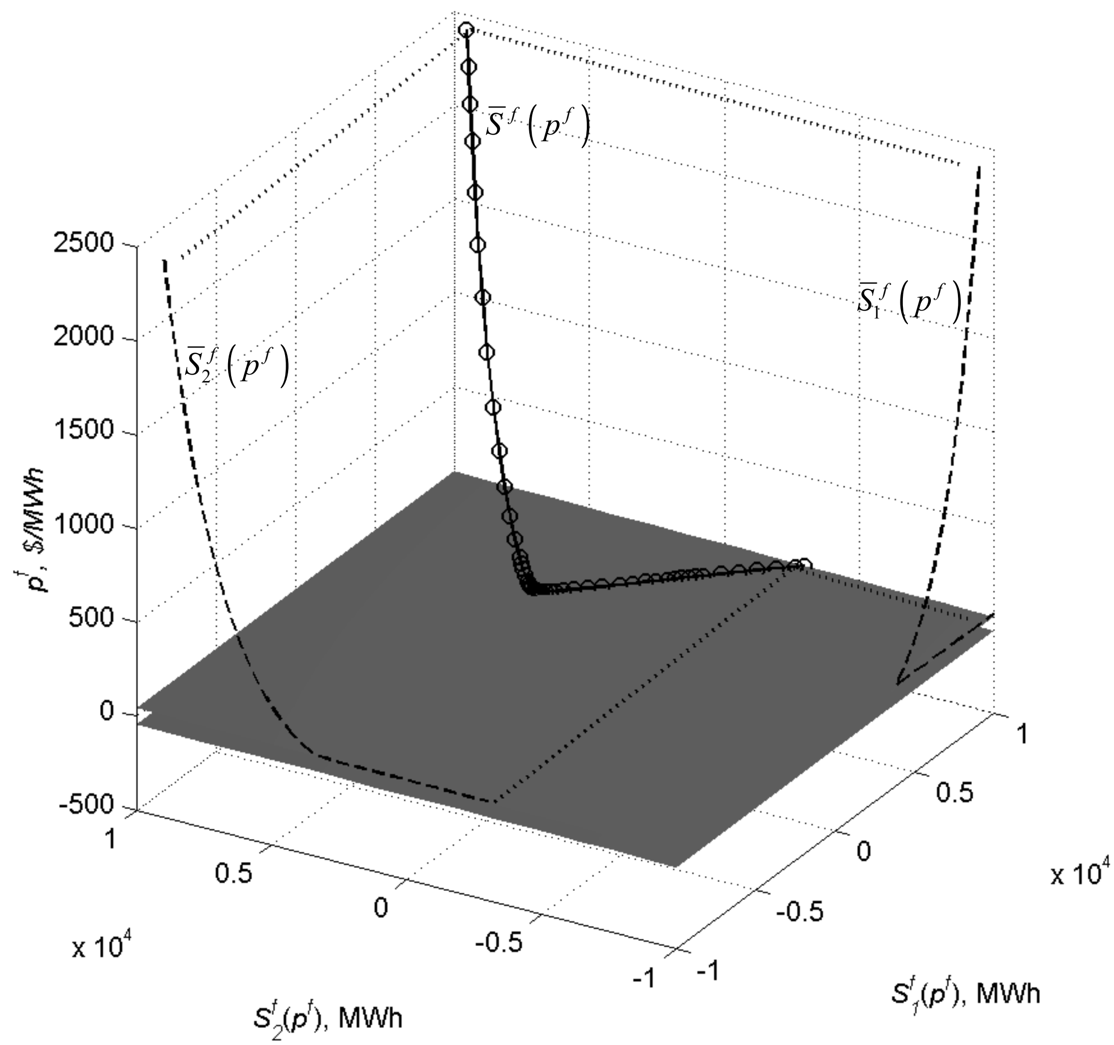

FIGURE 7.7: AN SF TRAJECTORY $\bar{S}^f\left(p^f\right)$ (SOLID LINE, MARKED WITH "O") IN THE UPPER PARTITION OF THE PHASE SPACE, ITS PLANAR PROJECTIONS—THE SFS $\bar{S}_1^f\left(p^f\right)$ AND $\bar{S}_2^f\left(p^f\right)$ (DASHED LINES)—FOR FIRMS 1 AND 2, AND THE ∞-LOCUS (BLACK SURFACE)

Figure 7.8 below plots the projections from Figure 7.7—the SFs $\bar{S}_1^f\left(p^f\right)$ and $\bar{S}_2^f\left(p^f\right)$—in a common price-quantity plane.

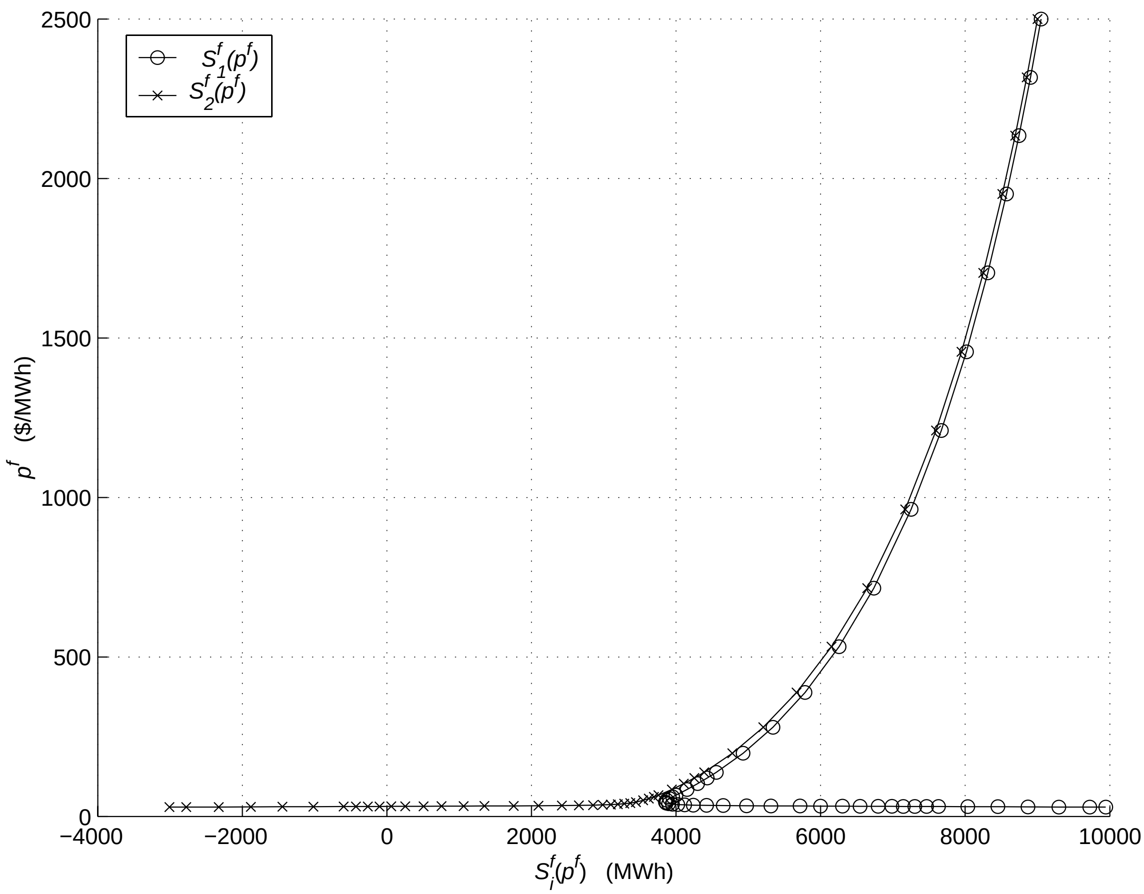

FIGURE 7.8: THE SFS $\bar{S}_1^f\left(p^f\right)$ AND $\bar{S}_2^f\left(p^f\right)$ OBTAINED FROM PLANAR PROJECTIONS OF THE SF TRAJECTORY $\bar{S}^f\left(p^f\right)$ IN FIGURE 7.7, PLOTTED IN A COMMON p^f-$\bar{q}_i^f$ PLANE

From Figure 7.8, we see that, for the particular SFs depicted, $\bar{S}_2^f\left(p^f\right)$ is everywhere strictly increasing, and $\bar{S}_1^f\left(p^f\right)$ is strictly increasing at all but the lowest prices (i.e., strictly increasing for $p^f \geq \$44/\text{MWh}$, approximately). As the figure suggests, whether a particular SF slopes upward depends on the chosen price range for integration as well as on the initial condition, for a given parameter vector Θ (whereby $\Theta = \Theta^{base}$, in this case).

We now characterize more closely the set of points in the upper partition at which the trajectory $\bar{S}^f\left(p^f\right)$ is such that the SFs $\bar{S}_i^f\left(p^f\right)$ are strictly increasing. To do so, some additional terminology will be useful. Namely, we define a *region* to be an open connected set of points within any given partition over which the signs of both SF slopes $\bar{S}_1^{f\prime}\left(p^f\right)$ and $\bar{S}_2^{f\prime}\left(p^f\right)$ are invariant. From the definitions of the 0_i-loci in Table 7.1 in the previous subsection, it is clear that within the given partition, the 0_i-loci constitute the boundaries of the regions. In other words, within each partition, we will have several regions, demarcated by the 0_i-loci and the ∞-locus (recall Figure 7.4 above).

While we may further subdivide each partition of the phase space into regions, our focus in this subsection is on the upper partition alone; we consider now the constituent regions of this partition. To this end, Figure 7.9 below reintroduces both of the 0_i-loci (as shown, for example, in Figure 7.4 above), emphasizing via choice of axis scales the portions of these loci lying in the upper partition in a neighborhood of the origin. The figure depicts the ∞-locus, as well; we may think of the ∞-locus as the lower boundary (i.e., in the $-p^f$ direction) of the upper partition. Consistent with the previous subsection's graphical conventions, Figure 7.9 portrays the ∞-locus in black, the 0_1-locus as a triangular mesh, and the 0_2-locus in gray.

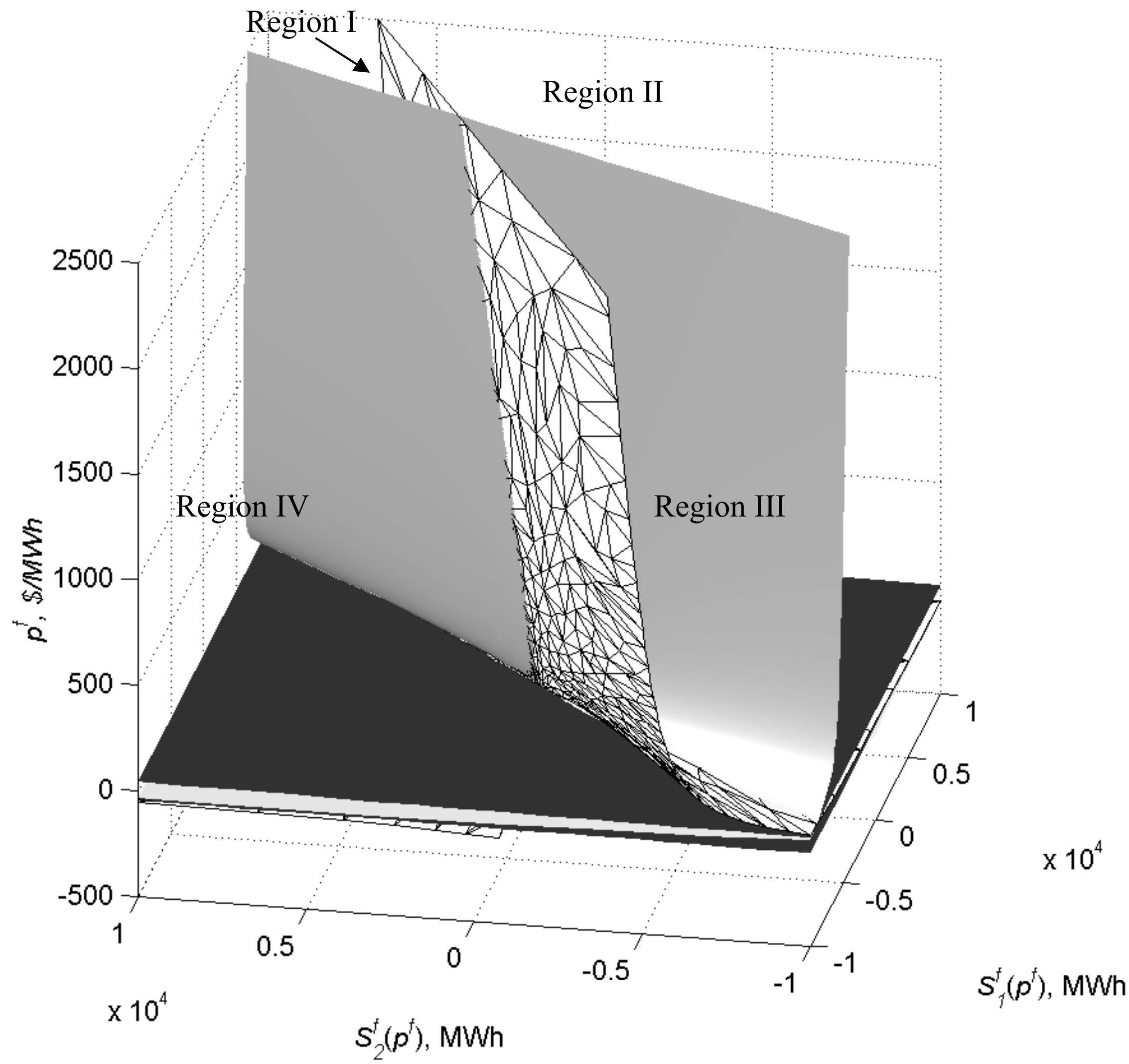

FIGURE 7.9: THE UPPER PARTITION COMPRISES REGIONS I–IV (SEE TEXT BELOW FOR DETAILS), BOUNDED BY THE ∞-LOCUS (IN BLACK), THE 0_1-LOCUS (A TRIANGULAR MESH), AND THE 0_2-LOCUS (IN GRAY)

Figure 7.9 also depicts *four regions* in the upper partition, numbered I–IV, delimited by the various loci and defined with respect to the signs of the SF slopes $\bar{S}_i^{f\prime}\left(p^f\right)$ as follows:

- Region I: $\bar{S}_1^{f\prime}\left(p^f\right)>0$, $\bar{S}_2^{f\prime}\left(p^f\right)>0$
- Region II: $\bar{S}_1^{f\prime}\left(p^f\right)<0$, $\bar{S}_2^{f\prime}\left(p^f\right)>0$
- Region III: $\bar{S}_1^{f\prime}\left(p^f\right)<0$, $\bar{S}_2^{f\prime}\left(p^f\right)<0$
- Region IV: $\bar{S}_1^{f\prime}\left(p^f\right)>0$, $\bar{S}_2^{f\prime}\left(p^f\right)<0$

For simplicity, Figure 7.9 does not depict SF trajectories in addition to the various loci, and also does not attempt to depict or label the various regions within the middle or lower partitions. As discussed above, the distinctions among the four regions labeled in Figure 7.9 follow from the definitions of the 0_i-loci (recall, e.g., Table 7.1). Since we seek strictly increasing SFs for both firms, we can identify *Region I* from the above definitions as that portion of the phase space that is most of interest for the multi-settlement SFE model.

For expository purposes, however, we consider first a variety of trajectories having, in general, both positively- and negatively-sloped portions over different price ranges. While we have not observed forward market SF trajectories in the upper partition that cross the ∞-locus, such trajectories can and do cross each of the 0_i-loci, as we demonstrate in this subsection. If a trajectory crosses the 0_1-locus (but not the 0_2-locus) at a particular point, for example, the sign of $\bar{S}_1^{f\prime}\left(p^f\right)$ changes at the crossing point, while the sign of $\bar{S}_2^{f\prime}\left(p^f\right)$ does not change.[267]

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[267] These sign changes are due, in turn, to changes in the signs of the numerators and denominators of the ratios on the right-hand sides of eqs. (7.40) and (7.41). In particular, from the definitions of the various loci, we have the following sign relationships for any trajectory:

The possibility that SFs can have both positively- and negatively-sloped sections is not novel in the SFE literature. As examples, we may cite two models of spot market SF competition for which the equilibrium SFs slope downward, at least for some prices. First, Klemperer and Meyer's (1989, 1254) model generates a continuum of SFs as their "Figure 1"—reproduced below as Figure 7.10—depicts.[268] In this continuum of SFs (see Figure 7.10 below), the sections of the SFs that (1) lie above the $f(p,S)=0$ locus, or (2) lie below the $f(p,S)=\infty$ locus, are decreasing in price *p*. Conversely, the sections of the SFs lying between these two loci are increasing in *p*. As noted in subsection 1.5.1 above, a second instance of downward-sloping SFs in the literature is Bolle (1992, 99), who finds SFs (in his "Model B") that are everywhere downward-sloping functions of price.

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1. If the trajectory crosses the 0_1-locus, the numerator $\overline{S}^{f++}\left(p^f\right)^{\top} \mathcal{Q}_{23}\overline{S}^{f++}\left(p^f\right)$ on the right-hand side of eq. (7.40) changes sign.
2. If the trajectory crosses the 0_2-locus, the numerator $\overline{S}^{f++}\left(p^f\right)^{\top} \mathcal{Q}_{13}\overline{S}^{f++}\left(p^f\right)$ on the right-hand side of eq. (7.41) changes sign.
3. For two trajectories on either side of the ∞-locus (and separated only by this locus), the denominators $\overline{S}^{f++}\left(p^f\right)^{\top} \mathcal{Q}_{12}\overline{S}^{f++}\left(p^f\right)$ and $\overline{S}^{f++}\left(p^f\right)^{\top} \mathcal{Q}_{21}\overline{S}^{f++}\left(p^f\right)$ on the right-hand sides of eqs. (7.40) and (7.41) have opposite signs.

Moreover, within the phase space's upper partition that we study here, the denominators of the ratios on the right-hand sides of eqs. (7.40) and (7.41) are negative and positive, respectively; the signs of the numerators of these ratios then determine the signs of the slopes $\overline{S}_1^{f\prime}\left(p^f\right)$ and $\overline{S}_2^{f\prime}\left(p^f\right)$.

[268] Assuming that the shock to the demand function in Klemperer and Meyer's model has finite support.

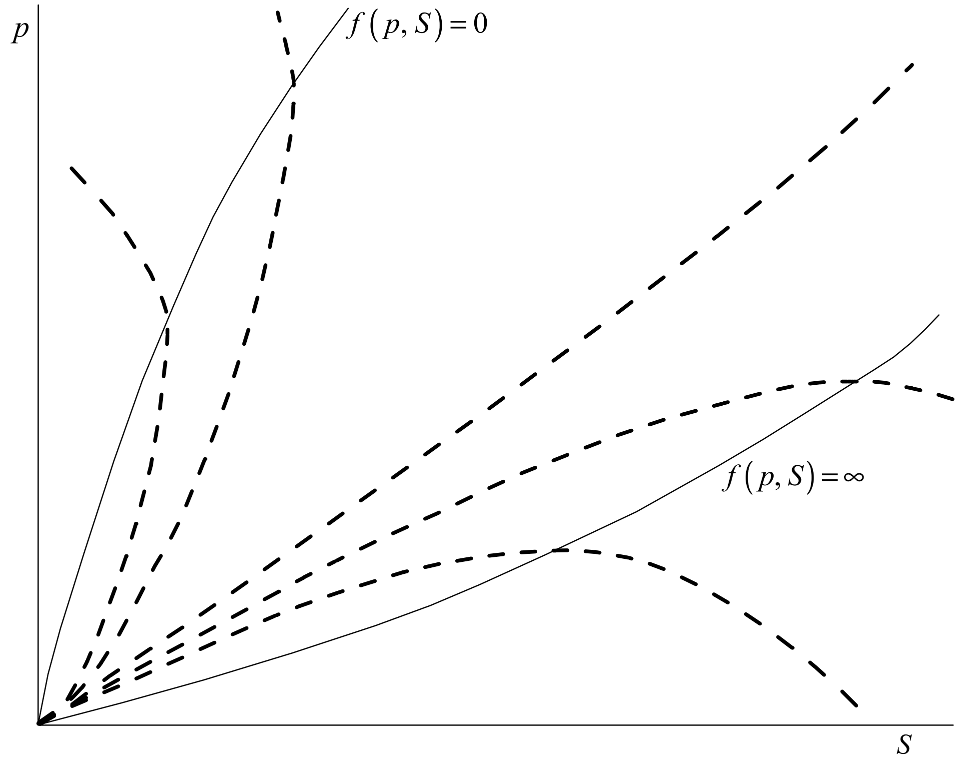

FIGURE 7.10: KLEMPERER AND MEYER'S (1989, 1254) FIGURE 1 DEPICTING THE $f(p,S)=0$ AND $f(p,S)=\infty$ LOCI (SOLID LINES), AND SUPPLY FUNCTIONS (DASHED LINES) SATISFYING THE DIFFERENTIAL EQUATION

$$f(p,S) \equiv \frac{S}{p-C'(S)} + D'(p)$$

AND HAVING BOTH POSITIVELY- AND NEGATIVELY-SLOPED SECTIONS

To provide additional insight into the qualitative behavior of solutions to the ODE system for the forward market, the following series of figures depicts three different examples of trajectories (for a variety of initial conditions) inhabiting the upper partition. We show how these trajectories pass among the various regions in this partition over the chosen range of integration, and examine how the path of each trajectory corresponds to changes in the slopes of each firm's SFs. As with Klemperer and Meyer (1989) and Bolle (1992) for the spot market, the examples presented below indicate for the forward market that—depending on equilibrium selection and the price domain considered—non-negative constraints on SF slopes could well be binding in equilibrium. This suggests, further, that such SF slope constraints could be potentially important considerations in market design.

Figure 7.11 below portrays an SF trajectory in the price range $p^f \in [100, 2{,}500]$ \$/MWh. This trajectory begins in Region IV at $p^f = \$100/\text{MWh}$, passes through the 0_1-locus separating Regions III and IV at approximately $p^f = \$678/\text{MWh}$, and ends in Region III. Figure 7.11 is rotated so that the planes $p^f = \text{constant}$ are perpendicular to the page, to facilitate accurate reading of the price p^f for points along the SF trajectory.[269] Finally, note that Region I is hidden on the "other side" of this figure, and is not labeled.

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[269] An unfortunate side effect of the perspective of Figure 7.11 is that the $\overline{q}_1^f$ and $\overline{q}_2^f$ axes are collinear in Figure 7.11, although these axes are, of course, perpendicular in $\mathbb{R}^3$. To clarify the perhaps confusing labeling of these axes, the axes share the lower limit of "–1 x 10^4" MWh at the bottom of the figure. The $\overline{q}_1^f$ axis extends to the right from this point, while the $\overline{q}_2^f$ axis extends to the left (in each case, to an upper limit of "1 x 10^4" MWh).

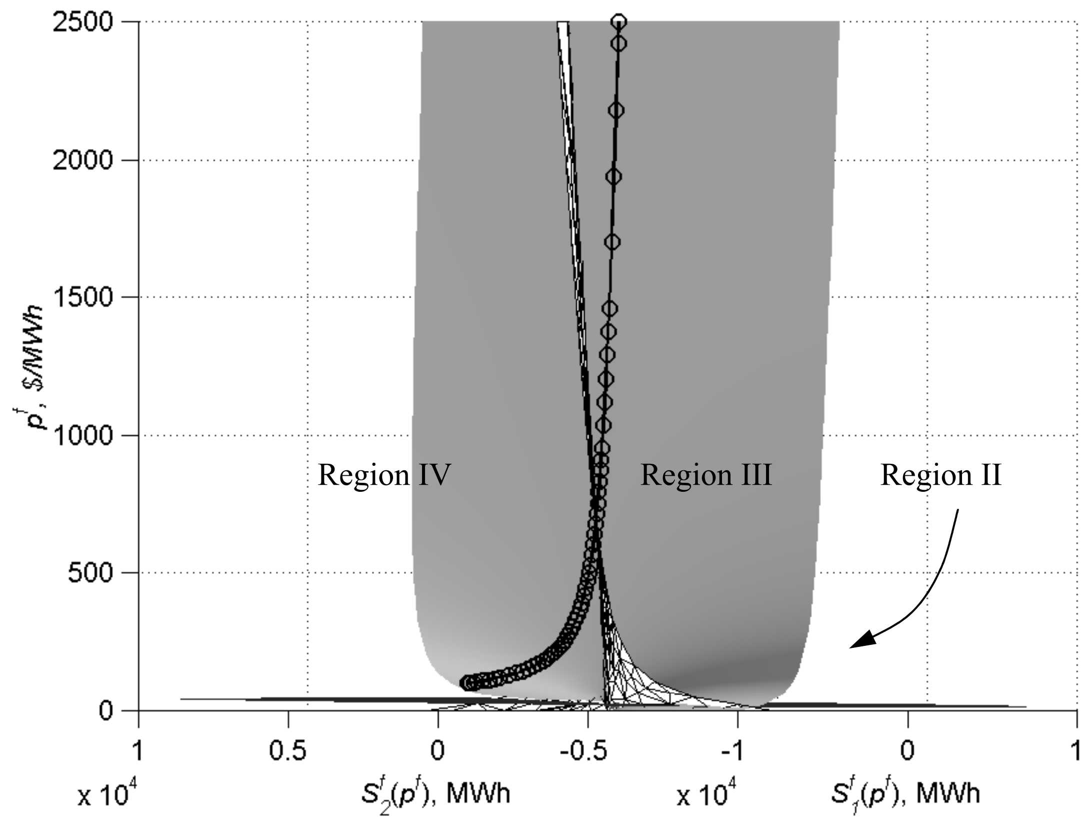

FIGURE 7.11: AN SF TRAJECTORY BEGINNING IN REGION IV AT $p^f = \$100/\text{MWh}$, PASSING THROUGH THE 0_1-LOCUS AT APPROXIMATELY $p^f = \$678/\text{MWh}$, AND ENDING IN REGION III AT $p^f = \$2{,}500/\text{MWh}$

Figure 7.12 below plots the projections of Figure 7.11's SF trajectory as the two firms' SFs in a common price-quantity plane.

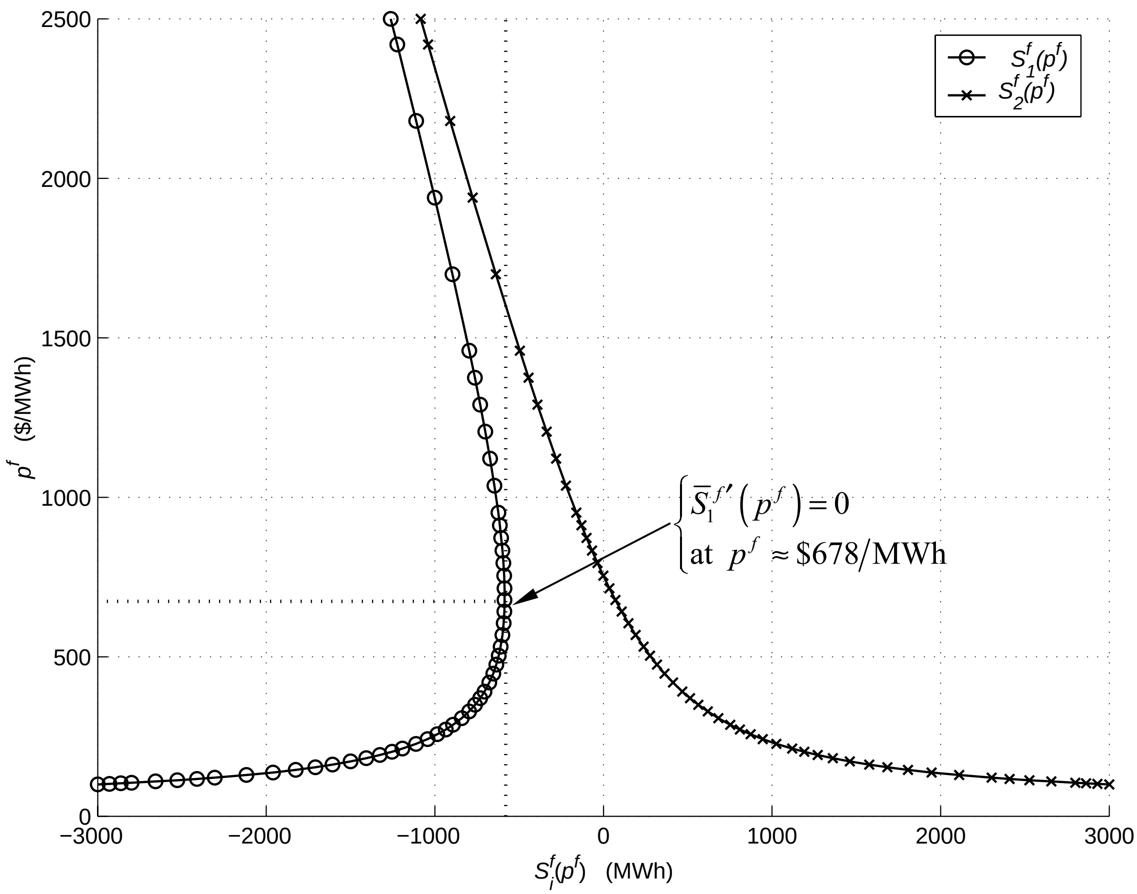

FIGURE 7.12: THE SFs $\bar{S}_1^f\left(p^f\right)$ AND $\bar{S}_2^f\left(p^f\right)$ CORRESPONDING TO THE SF TRAJECTORY $\bar{S}^f\left(p^f\right)$ IN FIGURE 7.11

Note, in particular, that the point at which the trajectory in Figure 7.11 passes through the 0_1-locus coincides with the point in Figure 7.12 at which $\bar{S}_1^f\left(p^f\right)$ bends back through the vertical ($\bar{S}_1^{f\prime}\left(p^f\right)=0$ at $p^f \approx \$678/\text{MWh}$), and becomes downward-sloping.

We present another example portraying a trajectory on the "other side" of the upper partition. Figure 7.13 below depicts an SF trajectory in the price range

$p^f \in [100, 2{,}500]$ \$/MWh. This trajectory begins in Region II at $p^f = \$100/\text{MWh}$, passes through the 0_2-locus separating Regions II and III at approximately $p^f = \$1{,}117/\text{MWh}$, and ends in Region III. Like Figure 7.11, Figure 7.13 is rotated so that the planes $p^f = \text{constant}$ are perpendicular to the page, to facilitate accurate reading of the price p^f.[270] Finally, note that Region I is hidden on the other side of this figure, and is not labeled.

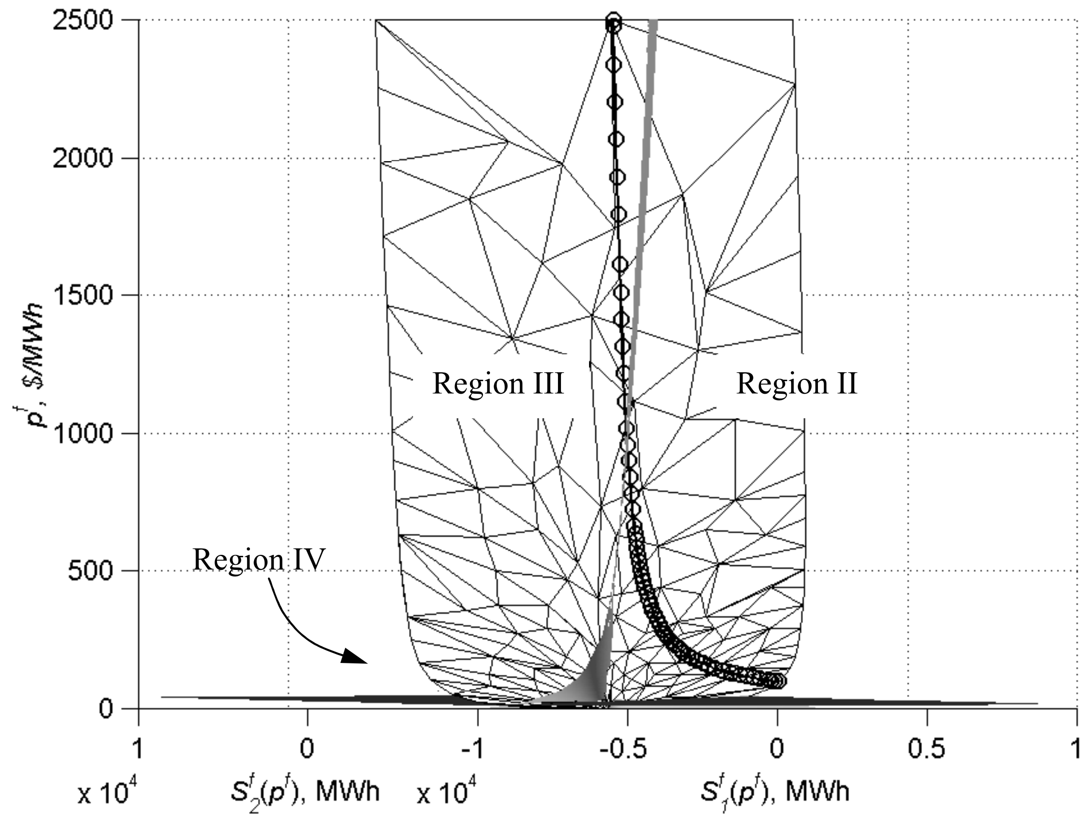

FIGURE 7.13: AN SF TRAJECTORY BEGINNING IN REGION II AT $p^f = \$100/\text{MWh}$, PASSING THROUGH THE 0_2-LOCUS AT APPROXIMATELY $p^f = \$1{,}117/\text{MWh}$, AND ENDING IN REGION III AT $p^f = \$2{,}500/\text{MWh}$

[270] Note 269 applies here, as well.

Figure 7.14 below plots the projections of Figure 7.13's SF trajectory as the two firms' SFs in a common price-quantity plane.

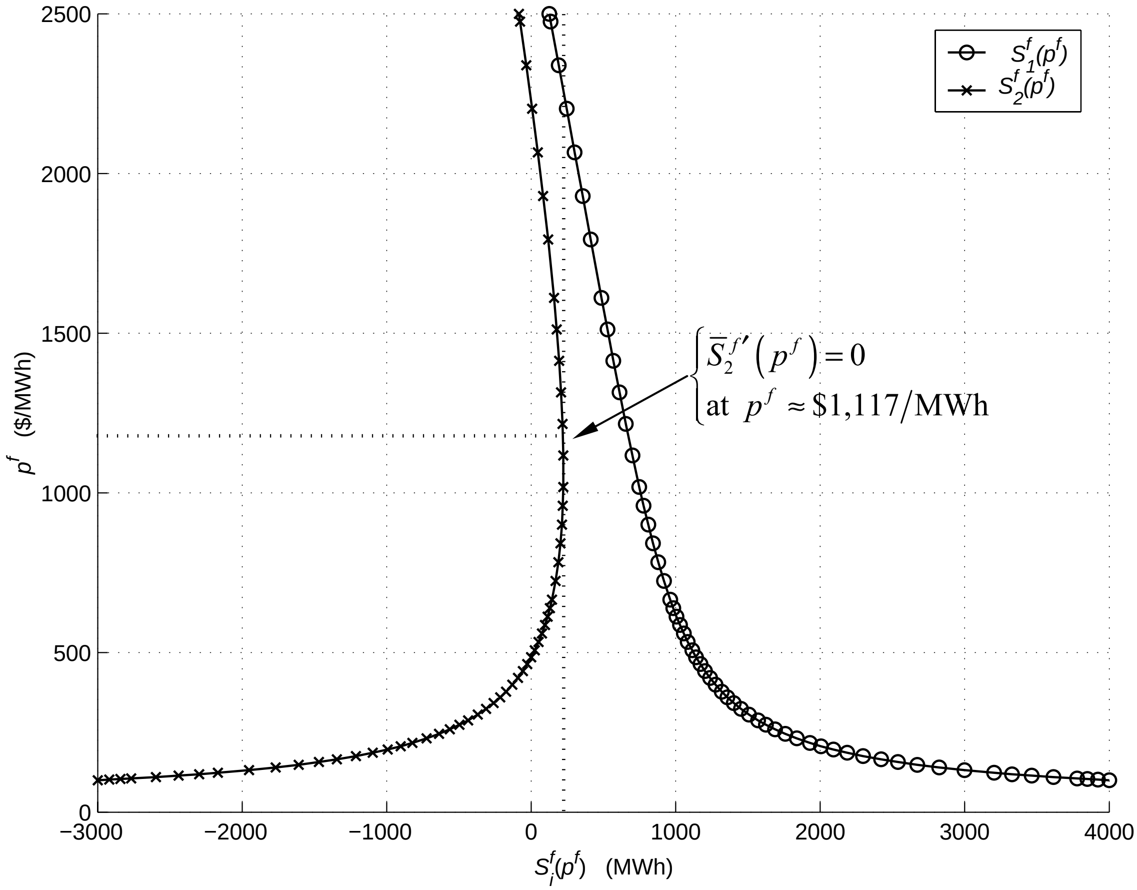

FIGURE 7.14: THE SFs $\bar{S}_1^f\left(p^f\right)$ AND $\bar{S}_2^f\left(p^f\right)$ CORRESPONDING TO THE SF TRAJECTORY $\bar{S}^f\left(p^f\right)$ IN FIGURE 7.13

The point at which the trajectory in Figure 7.13 passes through the 0_2-locus coincides with the point in Figure 7.14 at which $\bar{S}_2^f\left(p^f\right)$ bends back through the vertical ($\bar{S}_2^{f\prime}\left(p^f\right)=0$ at $p^f \approx \$1{,}117/\text{MWh}$), and becomes downward-sloping.

Turning now to this subsection's final pair of figures, Figure 7.15 below portrays an SF trajectory in the price range $p^f \in [500,\ 2{,}500]\ \$/\text{MWh}$. This trajectory begins in

Region I at $p^f = \$500/\text{MWh}$, first passes through the 0_1-locus separating Regions I and II at $p^f \approx \$1{,}830/\text{MWh}$, next passes through the 0_2-locus separating Regions II and III at $p^f \approx \$2{,}208/\text{MWh}$, and ends in Region III.

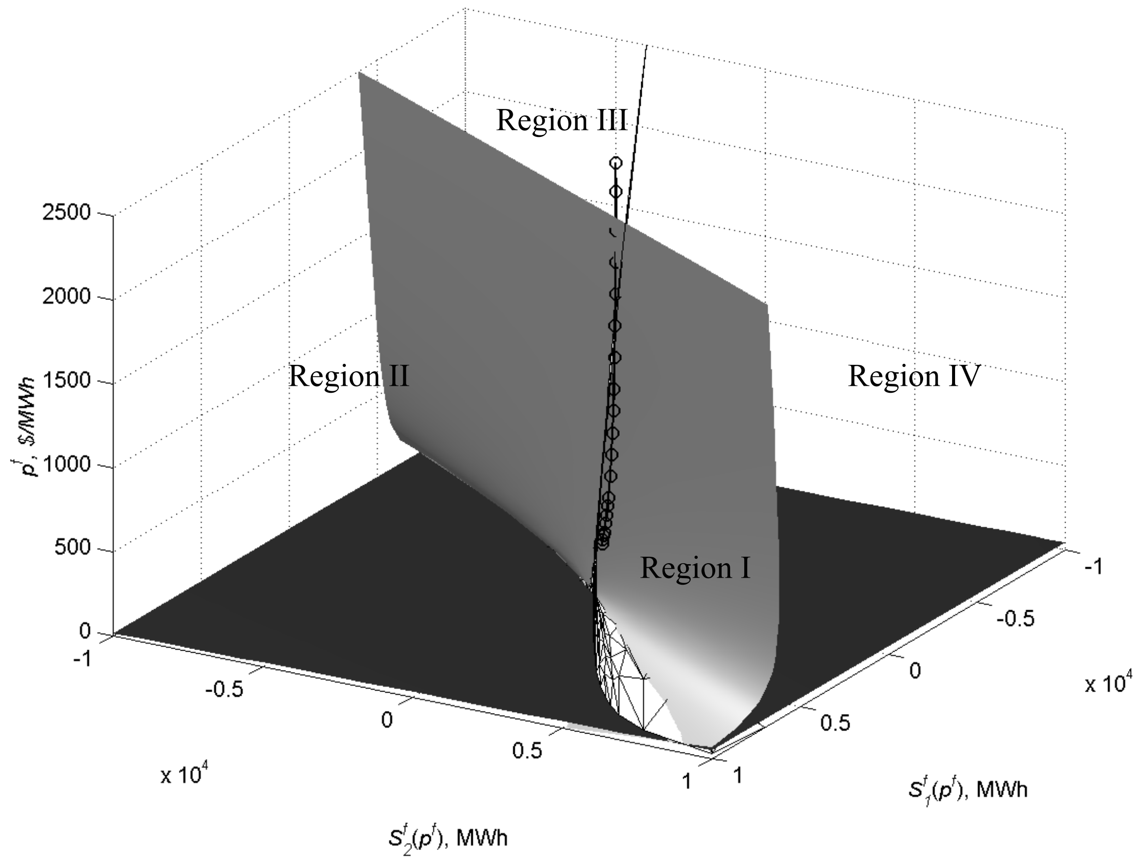

FIGURE 7.15: AN SF TRAJECTORY BEGINNING IN REGION I AT $p^f = \$500/\text{MWh}$, PASSING THROUGH THE 0_1-LOCUS AT $p^f \approx \$1{,}830/\text{MWh}$, PASSING THROUGH THE 0_2-LOCUS AT $p^f \approx \$2{,}208/\text{MWh}$, AND ENDING IN REGION III AT $p^f = \$2{,}500/\text{MWh}$

Figure 7.16 below plots the projections of Figure 7.15's SF trajectory as the two firms' SFs in a common price-quantity plane.

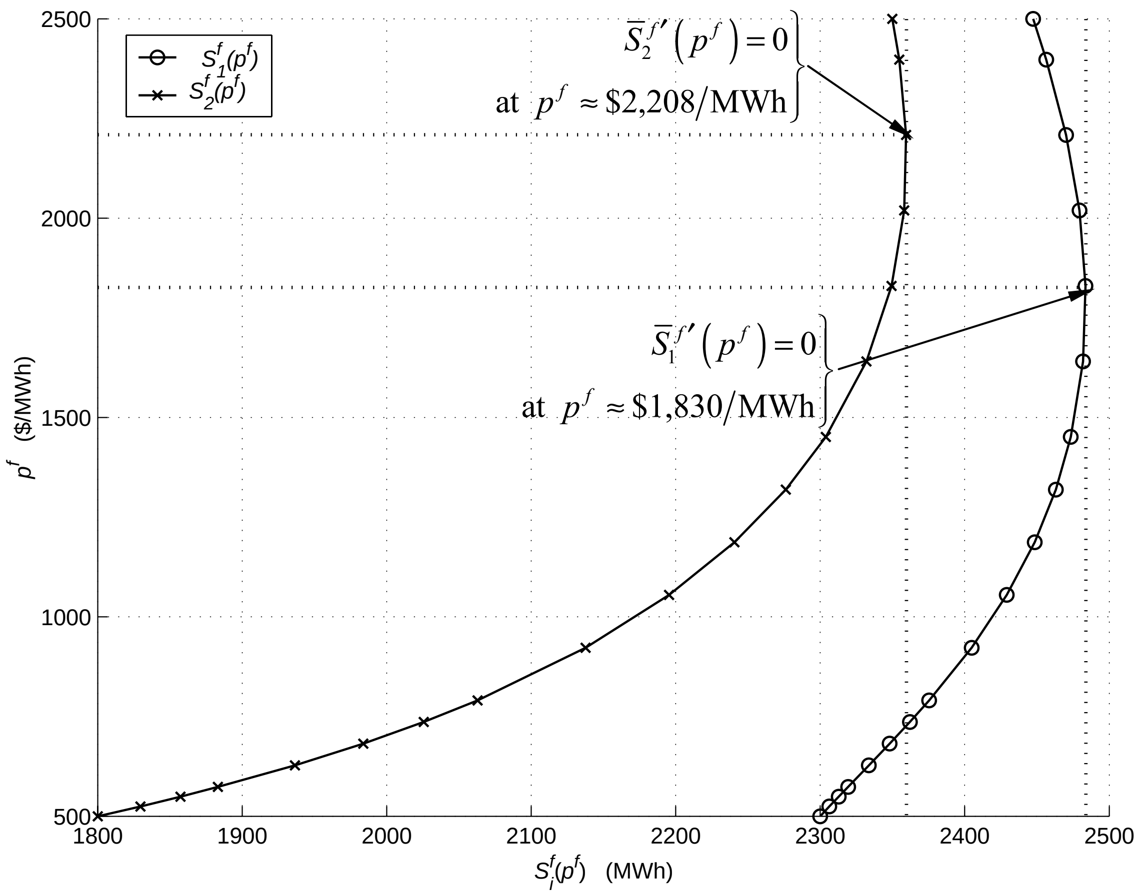

FIGURE 7.16: THE SFS $\overline{S}_1^f\left(p^f\right)$ AND $\overline{S}_2^f\left(p^f\right)$ CORRESPONDING TO THE SF TRAJECTORY $\overline{S}^f\left(p^f\right)$ IN FIGURE 7.15

The points at which the trajectory in Figure 7.15 passes through the 0_1- and 0_2-loci coincide with the points in Figure 7.16 at which $\overline{S}_1^f\left(p^f\right)$ and $\overline{S}_2^f\left(p^f\right)$ bend back through the vertical ($\overline{S}_1^{f\prime}\left(p^f\right)=0$ at $p^f \approx \$1{,}830/\text{MWh}$ and $\overline{S}_2^{f\prime}\left(p^f\right)=0$ at $p^f \approx \$2{,}208/\text{MWh}$, respectively), and become downward-sloping. Also, each firm's second-order condition for profit maximization is satisfied over the entire price range of the SFs in Figure 7.16.

The various "three-dimensional" figures above (i.e., Figure 7.11, Figure 7.13, and Figure 7.15, depicting the 0_i-loci and the ∞-locus) are the analogs to Klemperer and Meyer's (1989, 1254) Figure 1 for the forward market in the (asymmetric) multi-settlement SFE model. Figure 1 of KM's paper—redrawn as Figure 7.10 above—depicts various SFs solving the differential equation that characterizes (symmetric) spot market supply functions in their (single-market) model, along with the "$f(p,S)=0$" and "$f(p,S)=\infty$" loci analogous to the 0_i-loci and the ∞-locus discussed here. Figure 7.10 is suggestive of several characteristics of KM's SFs. For example—among other properties—all SFs pass through the origin (a singular point) with a common slope, and any nonsingular point has a unique SF passing through it. Such properties constitute the basis for KM's characterization of their SFs, proofs of existence, symmetry, and uniqueness of SFEs, and various comparative statics results. In our asymmetric multi-settlement SFE model, on the other hand, we prove existence and uniqueness of solutions (for a given initial condition) by appealing directly to properties of (nonsingular) systems of differential equations. Because we cannot solve the ODE system (7.40)–(7.42) explicitly, we are only able in the present work to conduct comparative statics analysis numerically (see section 7.6 below), rather than analytically, as KM did.

In future work, we may be able to characterize the SF trajectories—such as those depicted in the figures of this subsection—more precisely, and exploit their properties to prove additional results of greater generality than those documented here. Although the ODE system (7.40)–(7.42) is not analytically tractable, solutions to the system likely possess some properties that have not been explored here. For example, one conjecture based on numerical investigations is that if, as p^f increases, the trajectory enters Region III (of the upper partition) in which both SFs are downward-sloping, the trajectory remains in this region forever. Another conjecture is that, for p^f sufficiently large, both SFs are concave to the price axis. While these conjectures are presently unproven, future research could extend the catalog of such regularities, make them more precise, and possibly prove them analytically. The result would be a richer analytical characterization of the connected set of trajectories $\bar{S}^f\left(p^f\right)$, which might be helpful in sharpening and extending the generality of the comparative statics and other results presented in this work.

7.4.4 Price relationships across markets

We next investigate qualitatively the relationship of forward market and expected spot market equilibrium prices for a range of forward market outcomes. To do so, it will be useful to define analytically and graphically an additional construct for the forward market. Namely, denote as the *arbitrage plane* the set of forward market equilibrium points $\left[\bar{S}_1^f\left(p^f\right), \bar{S}_2^f\left(p^f\right), p^f\right]$ such that the forward market price p^f is equal to the conditional expectation of the spot market price, $\mathrm{E}\left(p^s \middle| p^f\right)$, given the price p^f. To characterize this locus, we set p^f equal to $\mathrm{E}\left(p^s \middle| p^f\right)$ in eq. (7.9) and solve for p^f, yielding the following equation of a plane in $\bar{q}_1^f$-$\bar{q}_2^f$-p^f space—the arbitrage plane, defined above:

$$p^f = \frac{\omega_a \left[(1-\phi_1)\bar{S}_1^f\left(p^f\right) + (1-\phi_2)\bar{S}_2^f\left(p^f\right) + \omega_b + \left(2-\gamma^s\omega_a\right)\left(\bar{v}_R - \frac{\sigma_{v_R^2, v_R}}{2\sigma_{v_R}^2}\right)\right]}{\left(2-\gamma^s\omega_a\right)}. \quad (7.50)$$

Figure 7.17 below depicts in $\bar{q}_1^f$-$\bar{q}_2^f$-p^f space the arbitrage plane (in gray), an SF trajectory (in the upper partition of the phase space), and the ∞-locus (in black) in a neighborhood of the origin.[271]

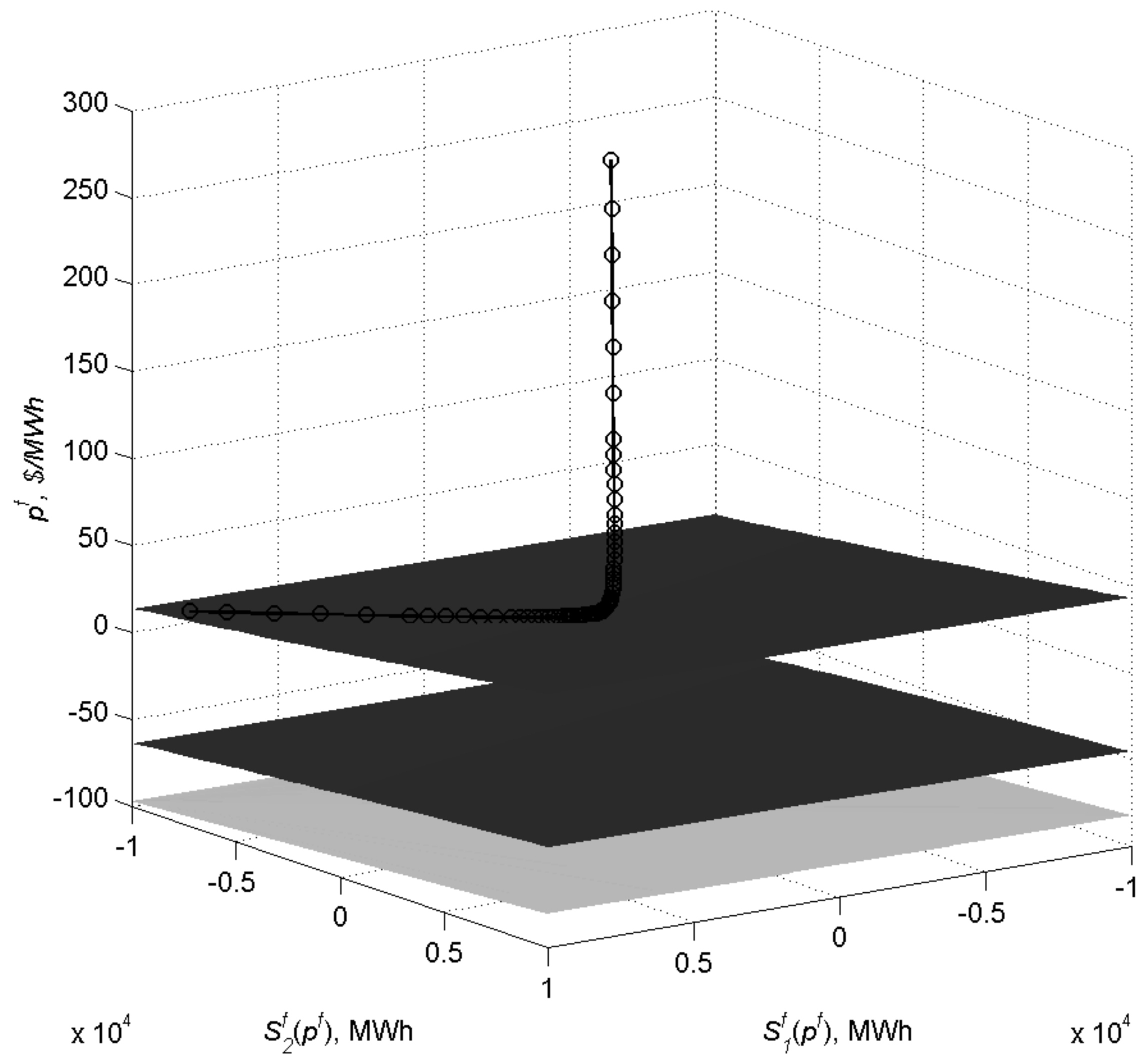

FIGURE 7.17: THE ARBITRAGE PLANE (GRAY SURFACE—SEE EQ. (7.50)), AN SF TRAJECTORY (SOLID LINE, MARKED WITH "O"), AND THE ∞-LOCUS (BLACK SURFACE) IN A NEIGHBORHOOD OF THE ORIGIN

[271] We plot the SF trajectory for $p^f \in [23.95, 300]$ \$/MWh and assuming base case parameter values Θ^{base}. For clarity, we do not plot the two 0_i-loci in the above figure.

From the definition of the arbitrage plane in eq. (7.50), we may infer the following relationships. For forward market equilibrium points in $\bar{q}_1^f$-$\bar{q}_2^f$-p^f space above the arbitrage plane, we have that $p^f > \mathrm{E}\left(p^s \middle| p^f\right)$, while for such points below the arbitrage plane, we have that $p^f < \mathrm{E}\left(p^s \middle| p^f\right)$. In the neighborhood of the subset of $\bar{q}_1^f$-$\bar{q}_2^f$-p^f space depicted in Figure 7.17, we see that the SF trajectory is everywhere above the ∞-locus (consistent with the trajectory's location in the upper partition). In contrast, the arbitrage plane is everywhere below the ∞-locus, situating it in the lower partition. These observations imply, further, that the SF trajectory in Figure 7.17 lies everywhere above the arbitrage plane, so that we conclude that the forward market equilibrium points comprising this SF trajectory are characterized by the inequality

$$p^f > \mathrm{E}\left(p^s \middle| p^f\right). \tag{7.51}$$

Moreover, inequality (7.51) applies along *any* SF trajectory lying in the upper partition that we may select within the neighborhood of the origin depicted in Figure 7.17.[272] If inequality (7.51) holds for all p^f along such a trajectory (as it will in a "moderately-sized" neighborhood of the origin—see note 272), we have further that

$$\mathrm{E}\left(p^f\right) > \mathrm{E}\left(p^s\right). \tag{7.52}$$

The inequalities (7.51) and (7.52) indicate that a risk-neutral, SF-bidding supplier in the multi-settlement SFE model facing a downward-sloping forward market demand

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[272] At points in the phase space much more distant from the origin (i.e., for forward market quantities several orders of magnitude larger), the arbitrage plane may cross one or both nappes of the ∞-locus, and hence leave the lower partition of the phase space. In such a case, the inequality (7.51) may not hold at all points along SF trajectories in the upper partition.

function[273] will *not* act so as to "equalize the market prices" in either the sense $p^f = \mathrm{E}\left(p^s \middle| p^f\right)$ or $\mathrm{E}\left(p^f\right) = \mathrm{E}\left(p^s\right)$. We may conclude that in this model, perfect inter-market price arbitrage is not a necessary implication of profit maximization. For such arbitrage to obtain would require, for example, that we introduce risk-neutral traders (with no trading limits) into the model.[274]

Inequalities (7.51) and (7.52) are natural results for our base case trajectory given the assumptions of the multi-settlement SFE model. To see why, recall that in chapter 6, we assume that the representative consumer *R* is risk averse. Accordingly, this consumer *R* pays a risk premium to the (risk-neutral) suppliers in the forward market, leading to forward market prices p^f in excess of conditional expected spot market prices $\mathrm{E}\left(p^s \middle| p^f\right)$.

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[273] And abstracting, as in the multi-settlement SFE model, from any risk-neutral traders.

[274] Alternatively, without introducing additional agents, it appears that permitting the representative consumer *R* to become progressively less risk averse tends to equate $\mathrm{E}\left(p^s \middle| p^f\right)$ and p^f (and tends to make the forward market demand function $D^f\left(p^f, \varepsilon_0^f\right)$ approach the horizontal). That is, preliminary numerical simulations for base case parameter values suggest that $\lim_{\lambda_R \to 0^+} \left[p^f - \mathrm{E}\left(p^s \middle| p^f\right)\right] = 0$. While consistent with intuition, exploring the generality of this numerical result is left for future work.

7.4.5 *Equilibrium in the forward market*

To conclude the qualitative graphical analysis of forward market competition in this section, we illustrate in Figure 7.18 below the determination of the forward market equilibrium price $p^f = p^{f*}\left(\varepsilon_0^f\right)$ via the intersection of forward market aggregate supply and demand.

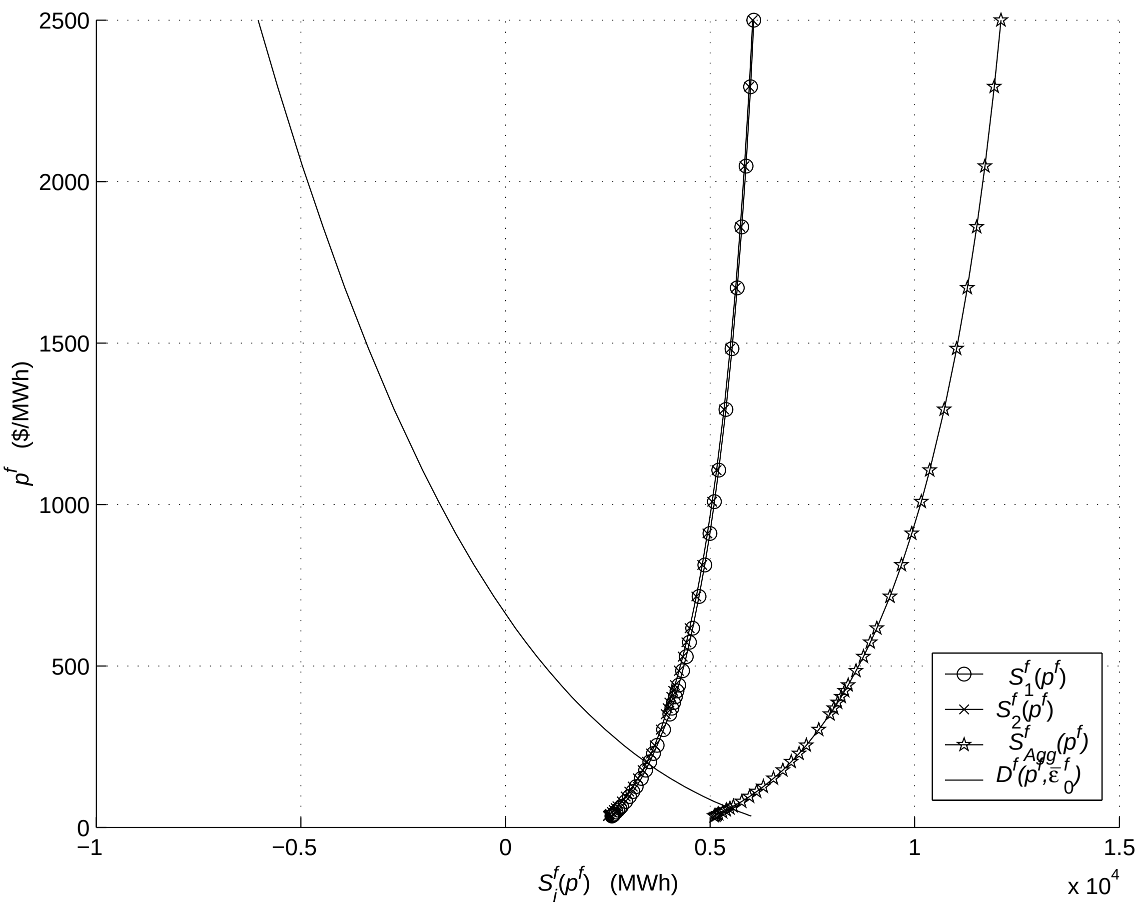

FIGURE 7.18: FORWARD MARKET EQUILIBRIUM FOR EXAMPLE SUPPLY FUNCTIONS AND MEAN DEMAND SHOCK $\bar{\varepsilon}_0^f \equiv \mathrm{E}\left(\varepsilon_0^f\right) = 6{,}008\,\mathrm{MWh}$, YIELDING AN EQUILIBRIUM PRICE $p^f \equiv p^{f*}\left(\bar{\varepsilon}_0^f\right) = \$59.42/\mathrm{MWh}$ AND AGGREGATE QUANTITY $\bar{q}_{Agg}^f \equiv \bar{S}_{Agg}^f\left(p^f\right) = 5{,}488\,\mathrm{MWh}$

Figure 7.18 is analogous to Figure 5.3 for the spot market. In the figure above, aggregate forward market supply $\overline{S}^f_{Agg}\left(p^f\right)$ intersects forward market demand $D^f\left(p^f, \overline{\varepsilon}^f_0\right)$, given the mean forward market demand shock $\overline{\varepsilon}^f_0 \equiv \mathrm{E}\left(\varepsilon^f_0\right)$ corresponding to Θ^{base}. Naturally, this intersection defines the equilibrium for the forward market, at which the equilibrium price is $p^f \equiv p^{f*}\left(\overline{\varepsilon}^f_0\right) = \$59.42/\text{MWh}$ and the aggregate quantity is $\overline{q}^f_{Agg} \equiv \overline{S}^f_{Agg}\left(p^f\right)$ $= 5{,}488\ \text{MWh}$.[275]

7.4.6 Equilibrium solution of the differential equation system

To conclude the discussion of the qualitative properties of system (7.40)–(7.42), we investigate the existence of an *equilibrium solution* to this differential equation system. First, we distinguish this new concept of an equilibrium solution of a differential equation system from the notion of supply function equilibrium. Recall that Table 3.1 defined a multi-settlement *supply function equilibrium* as sequence of equilibrium (optimal) SFs $\left\{\overline{S}^f_i\left(p^f\right), \overline{\Sigma}^s_i\left(p^s;\bullet\right)\right\}$, one for each market.[276] We also imposed the restriction that these SFs must be strictly increasing in their price arguments. Now, contrast supply function equilibria with the concept of an *equilibrium*—or *steady-state—solution* of a differential equation (DE) system. For brevity, we refer to this concept as a *DE equilibrium*. Many applications of differential equations to dynamic systems use time as the independent

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[275] We chose the initial quantities for the forward market SFs in Figure 7.18 so that these SFs slope upward, and also to ensure reasonable magnitudes for the forward market quantities, given the mean forward market demand shock $\overline{\varepsilon}^f_0$. Section 7.6 presents a systematic procedure for selecting a particular pair of forward market SFs from the connected set of SFEs.

[276] Where we later found that $\overline{\Sigma}^s_i\left(p^s;\bullet\right) \equiv \overline{\Sigma}^s_i\left(p^s; \overline{q}^f_i, \overline{q}^f_j\right)$.

variable, rather than price, as in the present model. In time-dependent problems, the use of "steady-state" as a synonym for "equilibrium" reflects the temporal nature of the concept of DE equilibrium. Namely, at a DE equilibrium of a time-dependent problem, values of dependent variables are *fixed* for all time t beginning with the initial time.

We may generalize this characterization of DE equilibrium for our problem of interest—a static problem—in which price is the independent variable. Namely, a DE equilibrium of the system (7.40)–(7.42) for the forward market of the multi-settlement SFE model[277] would satisfy

$$\overline{S}^{f+\prime}\left(p^f\right)=0 \tag{7.53}$$

for all $p^f \in \left[p^{f,0},\infty\right)$ (assuming upward integration from an initial price $p^{f,0}$). From the third component of the vector equation (7.53), a DE equilibrium must satisfy

$$\overline{S}_3^{f\prime}\left(p^f\right)=0 \tag{7.54}$$

for all $p^f \in \left[p^{f,0},\infty\right)$. Recalling the definition $\overline{S}_3^f\left(p^f\right)\equiv p^f$, however, eq. (7.42) held that $\overline{S}_3^{f\prime}\left(p^f\right)=1$ for all p^f. Equation (7.42) thereby contradicts eq. (7.54) and we conclude that the system (7.40)–(7.42) has *no* DE equilibria.

This result is not surprising, since DE equilibria "are not usually associated with non-autonomous equations although they can occur" (Jordan and Smith 1999, 6). Recall

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[277] We need consider here only the forward market since, in the simplified affine example for the spot market, a function $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ such that $\overline{\Sigma}_i^{s\prime}\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)=0$ at all p^s (i.e., a vertical spot market SF) exists only in the limit as $c_i\to\infty$. Thus, we conclude that there is no DE equilibrium in the spot market for finite parameter values.

that the original form of the system (7.40)–(7.42), eqs. (7.11) and (7.12), was indeed a non-autonomous equation, and converting it to an autonomous system via eq. (7.29) does not alter the presence (or absence, as is the case here) of DE equilibria. We examine here, in passing, the existence of DE equilibria for the system (7.40)–(7.42) since such equilibria and their properties are a common component of the qualitative analysis of differential equations. We emphasize that the nonexistence of DE equilibria for our system is inconsequential for our purposes.

What is of fundamental interest in the multi-settlement SFE model, naturally, are the SF trajectories and their dependence on initial conditions and parameter values. These relationships are the subject of comparative statics analysis in section 7.6 below. First, however, in the following section, we benchmark the discrete Excel model to ensure that it yields reasonable numerical results.

7.5 Benchmarking the discrete Excel model

This section describes how we benchmark the discrete Excel model using a representative multi-settlement market equilibrium. The purpose of this benchmarking procedure is to assign values to certain parameters of the multi-settlement SFE model that otherwise lack a plausible empirical basis for quantification. This procedure chooses these parameter values such that the mean equilibrium prices and quantities computed by the discrete Excel model agree, to the extent possible, with corresponding empirically-based reference values from the California market.[278] In this way, the benchmarking procedure

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[278] More fundamentally, it is the case that not all parameter values Θ produce equilibria having strictly increasing SFs in a subset of interest of $\overline{q}_1^f$-$\overline{q}_2^f$-p^f space. Therefore, even if our objective does not involve replicating certain empirical outcomes, simply requiring that SFs be strictly increasing in a given subset of this space places restrictions on the parameter vector Θ.

enhances the verisimilitude of solutions to the discrete Excel model in a sense that we make more precise below.

The benchmarking procedure comprises a lexicographic, two-step hierarchy. To summarize this procedure, the first benchmarking step produces numerically-computed expectations of *spot market* price and quantity that agree with corresponding empirical reference values from the California market. The second benchmarking step fixes these spot market expectations from the first step, and similarly computes expectations of *forward market* price and quantity that agree as closely as possible with the corresponding empirical reference data from California. Both benchmarking steps take as their basis a version of problem (7.43) from subsection 7.3.2 above. Each such step entails a revision of problem (7.43) in three respects:

1. First, we convert the parameters in problem (7.43) whose values are to be determined to *decision variables*. That is, we drop these parameters from the parameter vector Θ, yielding a "reduced" parameter vector. We add these same parameters to the problem's set of decision variables (i.e., along with the discretized SFs $\overline{S}_i^f\left(p^f\right)$), so that they become endogenous.[279] Subsections 7.5.1 and 7.5.2 below discuss the particular parameters to be converted to decision variables in each benchmarking step.

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[279] In each of the optimization problems presented in subsections 7.5.1 and 7.5.2 below (as well as in section 7.6's comparative statics analysis), we used the grid of forward market prices $p^f = 0, 250, 500, \ldots, 2{,}750$ \$/MWh to compute the discretized SFs $\overline{S}_i^f\left(p^f\right)$. That is, the discretized SFs each consist of eleven affine segments connecting twelve price-quantity pairs. While the discrete Excel model permits the user to adjust both the (uniform) step size and the range of prices p^f considered, the aforementioned grid of prices p^f yielded robust numerical results in each instance.

2. We use as the *objective function* in problem (7.43) the minimization (for each market, in turn) of the sum of squared proportional deviations of expected price and quantity from the corresponding empirical reference values. The choice of objective function constitutes an *equilibrium selection rule* for selecting a single forward market SF trajectory (assuming a unique solution for problem (7.43)) from the phase space of SF trajectories.
3. As appropriate, we introduce additional constraints into problem (7.43) that we call *benchmarking constraints*. These additional constraints equate certain expected equilibrium prices and quantities computed via the discrete Excel model with corresponding empirical reference values.

In the optimization problems that follow, $\mathrm{E}\left(p^s\right)$ denotes the expected spot market price and $\mathrm{E}\left(\overline{q}^s_{Agg}\right)$ the expected aggregate (equilibrium) spot market quantity. These expectations account for both forward and spot market uncertainty; that is, we compute these expectations with respect to the stochastic parameters η_R and ν_R (see section 6.5). We use the discrete Excel model to compute these expectations via discrete approximation of the joint cumulative distribution function of these parameters.

Subsections 7.5.1 and 7.5.2 below outline these two benchmarking steps in greater detail.

7.5.1 Benchmarking step 1 (spot market)

Recasting problem (7.43) in accordance with the discussion of paragraphs 1–3 above, we obtain—as step 1 of the benchmarking procedure—the following optimization problem:

$$\min_{\substack{\bar{S}_i^f\left(p^f\right)\ (discretized), \\ \bar{\eta}_R \geq 0,\, \sigma_{\eta_R}^2 \geq 0,\, \bar{\nu}_R \geq 0,\, \sigma_{\nu_R}^2 \geq 0}} \left[\frac{\mathrm{E}\left(p^s\right) - p_{empir}^{s,\,mean}}{p_{empir}^{s,\,mean}}\right]^2 + \left[\frac{\mathrm{E}\left(\bar{q}_{Agg}^s\right) - q_{empir}^{s,\,mean}}{q_{empir}^{s,\,mean}}\right]^2$$

$$s.t. \quad \text{Subgame-perfect Nash equilibrium in } \bar{\Sigma}_i^s \text{ and } \bar{S}_i^f \tag{7.55}$$

$$s.t. \quad \text{Parameters } \Theta^{(0)} \setminus \left(\bar{\eta}_R, \sigma_{\eta_R}^2, \bar{\nu}_R, \sigma_{\nu_R}^2\right).$$

Problem (7.55) converts the parameters $\bar{\eta}_R$, $\sigma_{\eta_R}^2$, $\bar{\nu}_R$, and $\sigma_{\nu_R}^2$—the means and variances of the stochastic parameters η_R and ν_R—in Θ to decision variables.[280] The objective function of this problem is the minimization of the sum of squared proportional deviations of the expected spot market price $\mathrm{E}\left(p^s\right)$ and expected aggregate spot market quantity $\mathrm{E}\left(\bar{q}_{Agg}^s\right)$ from the corresponding empirical reference values $p_{empir}^{s,\,mean}$ and $q_{empir}^{s,\,mean}$.[281] The notation $\Theta^{(0)} \setminus \left(\bar{\eta}_R, \sigma_{\eta_R}^2, \bar{\nu}_R, \sigma_{\nu_R}^2\right)$ represents the reduced parameter vector for problem (7.55).[282] The values chosen for the elements of this vector are, where possible, supported by empirical data.[283]

[280] We make this choice of additional decision variables as the result of experimenting with various formulations of the benchmarking procedure. The specification of problem (7.55) yields a feasible solution to this problem and, ultimately, reasonable base case values of all parameters, as discussed in subsection 7.5.3 below.

[281] See Appendix F.1.1 for details on these empirical values.

[282] That is, Θ with parameters $\bar{\eta}_R$, $\sigma_{\eta_R}^2$, $\bar{\nu}_R$, and $\sigma_{\nu_R}^2$ dropped.

[283] See Appendix F for details.

Problem (7.55)[284] yields optimal parameter values $\left(\bar{\eta}_R\right)^{(1)}$, $\left(\sigma^2_{\eta_R}\right)^{(1)}$, $\left(\bar{\nu}_R\right)^{(1)}$, and $\left(\sigma^2_{\nu_R}\right)^{(1)}$, which we collect along with other (fixed) parameters from $\Theta^{(0)} \setminus \left(\bar{\eta}_R, \sigma^2_{\eta_R}, \bar{\nu}_R, \sigma^2_{\nu_R}\right)$ in an intermediate parameter vector $\Theta^{(1)}$.[285] This problem also produces optimal forward market SFs $\bar{S}_i^f\left(p^f\right)\Big|^{(1)}$ for firms $i = 1, 2$.[286] Finally, the optimized objective function value of problem (7.55) ($\approx 1.14\text{e-}21$) is approximately zero, so that for practical purposes, we may consider the equalities $\mathrm{E}\left(p^s\right) = p^{s,\,mean}_{empir}$ and $\mathrm{E}\left(\bar{q}^s_{Agg}\right) = q^{s,\,mean}_{empir}$ to hold. This fact will be useful in Step 2 of the benchmarking procedure below.

7.5.2 Benchmarking step 2 (forward market)

Again recasting the general form of problem (7.43), we obtain—as step 2 of the benchmarking procedure—the following optimization problem:

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[284] We solve problem (7.55) without using automatic scaling, one of the Excel Solver's "Solver Options" (Microsoft Corporation 2001) (in Excel, see "Tools | Solver | Options"). Scaling may be useful in obtaining a feasible solution, particularly when the underlying matrices used by the Excel Solver to represent the optimization problem are poorly conditioned.

[285] The superscript "$^{(1)}$" denotes optimal values for benchmarking step 1 (problem (7.55)).

[286] Optimal, of course, only for problem (7.55).

$$\min_{\substack{\bar{S}_i^f\left(p^f\right)\ (discretized), \\ \bar{\eta}_R \geq 0, \sigma^2_{\eta_R} \geq 0, \bar{v}_R \geq 0, \sigma^2_{v_R} \geq 0, \\ e^s_{dem} \leq 0, \lambda_R \geq 0}} \left[\frac{\mathrm{E}\left(p^f\right) - p^{f,\,mean}_{empir}}{p^{f,\,mean}_{empir}}\right]^2 + \left[\frac{\mathrm{E}\left(\bar{q}^f_{Agg}\right) - q^{f,\,mean}_{empir}}{q^{f,\,mean}_{empir}}\right]^2$$

$$\begin{aligned} s.t.\quad & \text{Subgame-perfect Nash equilibrium in } \bar{\Sigma}^s_i \text{ and } \bar{S}^f_i \\ & s.t.\quad \mathrm{E}\left(p^s\right) = p^{s,\,mean}_{empir} \\ & \qquad\quad \mathrm{E}\left(\bar{q}^s_{Agg}\right) = q^{s,\,mean}_{empir} \\ & \qquad\quad \text{Parameters } \Theta^{(1)} \setminus \left(\bar{\eta}_R, \sigma^2_{\eta_R}, \bar{v}_R, \sigma^2_{v_R}, e^s_{dem}, \lambda_R\right). \end{aligned} \tag{7.56}$$

Problem (7.56) is related to problem (7.55) in four important ways. First, (7.56) adds two additional decision variables to those used in (7.55), namely, the spot market demand elasticity e^s_{dem}, and the representative consumer R's CARA parameter λ_R. We introduce these decision variables in problem (7.56) both to allow for maximum flexibility in improving (7.56)'s objective function, and because these parameters have a rather tenuous empirical basis, as Appendix F discusses.[287] Second, we use values in $\Theta^{(1)}$ and $\bar{S}^f_i\left(p^f\right)\Big|^{(1)}$ from problem (7.55) as initial values for decision variables and for the parameter values $\Theta^{(1)} \setminus \left(\bar{\eta}_R, \sigma^2_{\eta_R}, \bar{v}_R, \sigma^2_{v_R}, e^s_{dem}, \lambda_R\right)$ in problem (7.56). Third, problem (7.56)'s objective function minimizes the sum of squared proportional deviations of expected price $\mathrm{E}\left(p^f\right)$ and quantity $\mathrm{E}\left(\bar{q}^f_{Agg}\right)$ from the corresponding empirical reference values $p^{f,\,mean}_{empir}$ and $q^{f,\,mean}_{empir}$ for the forward market (rather than the spot market, as was the case in problem (7.55)).[288] Fourth and finally, we introduce the benchmarking

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[287] Note that the only parameters in $\Theta^{(1)}$ left fixed in problem (7.56) are the slopes c_i and intercepts c_{0i} of the firms' marginal cost functions.

[288] See Appendix F.2.1 for details on these empirical values.

constraints $\mathrm{E}\left(p^{s}\right)=p_{empir}^{s,\,mean}$ and $\mathrm{E}\left(\overline{q}_{Agg}^{s}\right)=q_{empir}^{s,\,mean}$ in problem (7.56). Recalling that these equalities held, effectively, in the solution to benchmarking step 1 (problem (7.55)), we may consider (7.56) a refinement of (7.55), that is, a refinement in the "direction" of a better fit to the empirical outcome in the forward market.

At an optimal solution to problem (7.56)[289]—benchmarking step 2—we have the optimal parameter values $\left(\overline{\eta}_{R}\right)^{(2)}$, $\left(\sigma_{\eta_R}^{2}\right)^{(2)}$, $\left(\overline{\nu}_{R}\right)^{(2)}$, $\left(\sigma_{\nu_R}^{2}\right)^{(2)}$, $\left(e_{dem}^{s}\right)^{(2)}$, and $\left(\lambda_{R}\right)^{(2)}$, which we collect along with other (fixed) parameters from $\Theta^{(1)}\setminus\left(\overline{\eta}_{R},\sigma_{\eta_R}^{2},\overline{\nu}_{R},\sigma_{\nu_R}^{2},e_{dem}^{s},\lambda_{R}\right)$ in (another) intermediate parameter vector $\Theta^{(2)}$. Problem (7.56) also yields optimal forward market SFs $\left.\overline{S}_{i}^{f}\left(p^{f}\right)\right|^{(2)}$ for firms $i=1,2$.

7.5.3 *Discussion*

The results of the benchmarking procedure outlined in this section include a vector $\Theta^{(2)}$ of parameter values that we use below as base case parameter values; that is, we set

$$\Theta^{base}=\Theta^{(2)}. \tag{7.57}$$

Equation (7.46) above gives values for the elements of the resulting vector Θ^{base}, values which are intuitively reasonable and consistent with a priori expectations. In addition, the benchmarking procedure yields a corresponding set of (discretized) forward market SFs $\left.\overline{S}_{i}^{f}\left(p^{f}\right)\right|^{(2)}$ for use as initial conditions in the comparative statics analysis of section 7.6. Moreover, this procedure also guarantees that the spot market benchmarking

[289] We solve problem (7.55) using automatic scaling in the Excel Solver (see note 284).

constraints $\mathrm{E}\left(p^s\right)=p_{empir}^{s,\,mean}$ and $\mathrm{E}\left(\overline{q}_{Agg}^s\right)=q_{empir}^{s,\,mean}$ from problem (7.56) still hold, essentially, in the base case problem analyzed in subsection 7.6.1 below.[290] In this sense, the benchmarking procedure enhances the verisimilitude of solutions to the discrete Excel model.

As noted at the outset of section 7.5, we may view the benchmarking procedure detailed in the foregoing subsections as a lexicographic approach to benchmarking the model. Under this approach, we first ensure that the spot market benchmarking constraints hold with equality. Then, for the forward market—while enforcing these spot market constraints—we seek the best possible agreement between the model and stylized reality.

7.6 Comparative statics analysis

This section describes the *comparative statics* of a discrete approximation to the ODE system (7.40)–(7.42) in which we investigate, in effect, the simultaneous perturbation of parameters and initial conditions for this system. This analysis entails, for each firm, a comparison of a "base case" SF (computed for base case parameter values) with a variety of "test case" SFs, each corresponding to a certain parameter perturbation.

We may decompose comparative statics analysis of the multi-settlement SFE model into several steps:

[290] That is, these equalities hold to within the convergence criterion chosen in the Excel Solver.

1. Choose a range of prices $\left[\underset{\smile}{p}^f, \widehat{p}^f\right]$ (and a step size Δp^f) over which to solve a version of problem (7.43), a difference equation approximation to the original ODE system (7.40)–(7.42).[291]
2. Fix two parameter vectors, a *base case vector* and a perturbed *test case vector*.
3. Choose an equilibrium selection rule, operationalized in problem (7.43) via the choice of objective function, for selecting a single forward market SF trajectory from the phase space of SF trajectories that solve this problem.[292]
4. Solve problem (7.43) twice using the chosen objective function, once for each parameter vector from step 2. The SF selected for each firm in this problem's solution will, in general, differ across the base case and test case. This implies, further, that each firm's initial quantity will also typically differ across the two cases.

We may then compare each firm's SF for the base case and the test case. In general, the direction in which a firm's SF is perturbed will not be uniform across all prices $p^f \in \left[\underset{\smile}{p}^f, \widehat{p}^f\right]$. That is, we observe not simply translations of the SFs, but also rotations and deformations of these functions, leading—after the perturbation—to higher quantities

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[291] For consistency, we choose the same grid of prices p^f (see note 279) as was used for section 7.5's benchmarking procedure to discretize the SFs $\overline{S}_i^f\left(p^f\right)$.

[292] Recall that problem (7.43)'s upper-level constraint "Subgame-perfect Nash equilibrium in $\overline{\Sigma}_i^s$ and $\overline{S}_i^f$" imposes the assumptions of the simplified affine example to solve the spot market problem. We require no equilibrium selection procedure for the spot market under these assumptions since this affine equilibrium is unique.

at some prices, and lower quantities at other prices.[293] The primary focus of this section's comparative statics analysis will be to document and explain the observed changes in firms' quantities under various perturbations to the system. Where helpful in developing intuition, we also discuss the effects on the SFs' slopes.[294]

The approach to comparative statics analysis outlined above incorporates both an equilibrium selection rule and a parameter perturbation in the base case and test case problems. This strategy combines two analytic techniques for differential equations that are usually treated separately in the literature: (1) *stability analysis*, which examines the effects of perturbation of *initial conditions*, and (2) *structural stability analysis*, which examines the effects of perturbation of *parameters*. Finally, since we cannot solve the ODE system (7.40)–(7.42) analytically to obtain an explicit expression for the trajectory $\overline{S}^f\left(p^f\right)$, we must conduct the comparative statics analysis numerically rather than analytically. Absent additional analytical results,[295] moreover, the comparative statics analysis is valid only locally, that is, for a particular base case parameter vector Θ^{base}.

The outline of this section is as follows. We use the discrete Excel model to compute the forward market SFs for the base case parameter vector in subsection 7.6.1, and then for various perturbed parameter vectors ("test cases") in subsection 7.6.2.

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[293] Subject, of course, to the constraints in problem (7.43) including, in particular, that both firms' SFs in both the base and test cases have non-negative slopes.

[294] In general, there exist comparative statics effects on the SFs' higher-order derivatives as well (curvatures, etc.), though it is naturally more difficult to find simple intuitive explanations underlying these more subtle effects.

[295] The qualitative analysis of subsection 7.4.3 concludes with some conjectures concerning more general properties of trajectories $\overline{S}^f\left(p^f\right)$. Further developing such conjectures, for example, may lead to more generally applicable comparative statics results.

Subsection 7.6.3 concludes, providing intuitive interpretations of the observed SF perturbations.

7.6.1 Computation of forward market SFs: Base case problem

We again reformulate problem (7.43) to obtain the following optimization problem (the "base case problem") for the comparative statics analysis:

$$
\begin{aligned}
\min_{\bar{S}_i^f\left(p^f\right)\ (discretized)} \quad & \left[\frac{\mathrm{E}\left(p^s\right)-p_{empir}^{s,\,mean}}{p_{empir}^{s,\,mean}}\right]^2+\left[\frac{\mathrm{E}\left(\bar{q}_{Agg}^s\right)-q_{empir}^{s,\,mean}}{q_{empir}^{s,\,mean}}\right]^2 \\
s.t. \quad & \text{Subgame-perfect Nash equilibrium in } \bar{\Sigma}_i^s \text{ and } \bar{S}_i^f \\
& s.t. \quad \bar{q}_i^s \geq 0 \\
& \qquad\quad \text{Parameters } \Theta^{base}.
\end{aligned}
\tag{7.58}
$$

Comparing problem (7.58) with problem (7.56), we note four important distinctions. First, we drop all parameters from (7.58)'s list of decision variables, retaining as decision variables only the discretized forward market SFs $\bar{S}_i^f\left(p^f\right),\ i=1,2$. Second, problem (7.58)'s objective function minimizes the sum of squared proportional deviations of expected price and quantity from the corresponding empirical reference values for the spot market. Third, because of problem (7.58)'s revised objective function, we drop the constraints $\mathrm{E}\left(p^s\right)=p_{empir}^{s,\,mean}$ and $\mathrm{E}\left(\bar{q}_{Agg}^s\right)=q_{empir}^{s,\,mean}$ used in (7.56). In addition, problem (7.58) fixes the base case parameter vector as $\Theta^{base}=\Theta^{(2)}$ (recalling eq. (7.57)) from problem (7.56). Fourth and finally, we introduce the constraints $\bar{q}_i^s \geq 0$ (for all states of the world) to preclude negative spot market quantities for suppliers from arising in the model. While the constraints $\bar{q}_i^s \geq 0$ do not bind in the optimal solution to the base case problem (7.58), we cannot, ex ante, rule out the possibility that they will bind in one or more of the test cases considered below. For consistency, we include these (non-binding) constraints here in the base case problem.[296]

The fact that $\overline{q}_i^s \geq 0$ does not bind in the base case problem reflects the positive forward market quantities $\overline{q}_i^f$ that result, in equilibrium, from the base case SFs $\overline{S}_i^f\left(p^f\right)$ (see Figure 7.19 below) selected via problem (7.58). Recalling the geometry of the spot market depicted in Figure 5.3, such quantities $\overline{q}_i^f > 0$ translate firms' spot market SFs (and hence the aggregate spot market SF) to the right, increasing the likelihood that spot market quantities $\overline{q}_i^s$ are positive in equilibrium.

Note that because the constraints $\mathrm{E}\left(p^s\right) = p_{empir}^{s,\,mean}$ and $\mathrm{E}\left(\overline{q}_{Agg}^s\right) = q_{empir}^{s,\,mean}$ were satisfied in (7.56), problem (7.58)'s objective function attains a minimum of essentially zero (given Excel's convergence criterion). As a consequence, problems (7.56) and (7.58) have the same optimal solution.[297] For the forward market, in contrast, we find the

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[296] Recall from subsection 3.1.5 that we defined each firm *i*'s spot market SF $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$, for simplicity, as having a range of $\mathbb{R}$, that is, $\overline{\Sigma}_i^s:\mathbb{R}^3 \to \mathbb{R}$. That subsection's construction of this SF $\overline{q}_i^s = \overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ relied on evaluating firm *i*'s marginal cost function $C_i'\left(\overline{q}_i^s\right)$ at each equilibrium quantity $\overline{q}_i^s$ resulting from $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$. We defined the function $C_i'\left(\overline{q}_i^s\right)$ only for $\overline{q}_i^s \geq 0$ (see subsection 3.1.8), however, so optimality of the function $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right)$ is not assured if $\overline{\Sigma}_i^s\left(p^s;\overline{q}_i^f,\overline{q}_j^f\right) = \overline{q}_i^s < 0$. Thus, constraining $\overline{q}_i^s$ to be non-negative in problem (7.58) (and in problem (7.61) below) ensures the optimality of the spot market quantities. The constraints $\overline{q}_i^s \geq 0$ imply that the duopoly suppliers are precluded from being net demanders in the spot market in this chapter's numerical examples.

Relaxing the constraints $\overline{q}_i^s \geq 0$ is possible, in principle, at the cost of introducing some additional structure into the model. Namely, absent these constraints, a supplier could become a net demander, and vice versa. Making such a scenario operational computationally would entail, for example, specifying a utility function for consumption on the part of suppliers, and conversely, specifying an electricity production technology for consumers.

[297] We solve problem (7.58) without using automatic scaling in the Excel Solver (see note 284).

following discrepancies between expected forward market price and quantity from the base case problem (7.58), on the one hand, and the corresponding forward market empirical reference values, on the other:

$$\mathrm{E}\left(p^f\right) - p_{empir}^{f,mean} = 912.95 - 26.60 = \$886.35/\mathrm{MWh} \tag{7.59}$$

$$\mathrm{E}\left(\overline{q}_{Agg}^f\right) - q_{empir}^{f,mean} = 4{,}983 - 4{,}033 = 950 \text{ MWh}\,. \tag{7.60}$$

The agreement between $\mathrm{E}\left(p^f\right)$ and $p_{empir}^{f,mean}$ is poor, but that between $\mathrm{E}\left(\overline{q}_{Agg}^f\right)$ and $q_{empir}^{f,mean}$ is reasonably close in relative terms: these quantities differ by only about 24%. We may view the differences noted in eqs. (7.59) and (7.60) above as a measure of the deviation of the multi-settlement SFE model from the actual market.

Figure 7.19 below plots the discretized forward market SFs solving problem (7.58), the comparative statics base case.

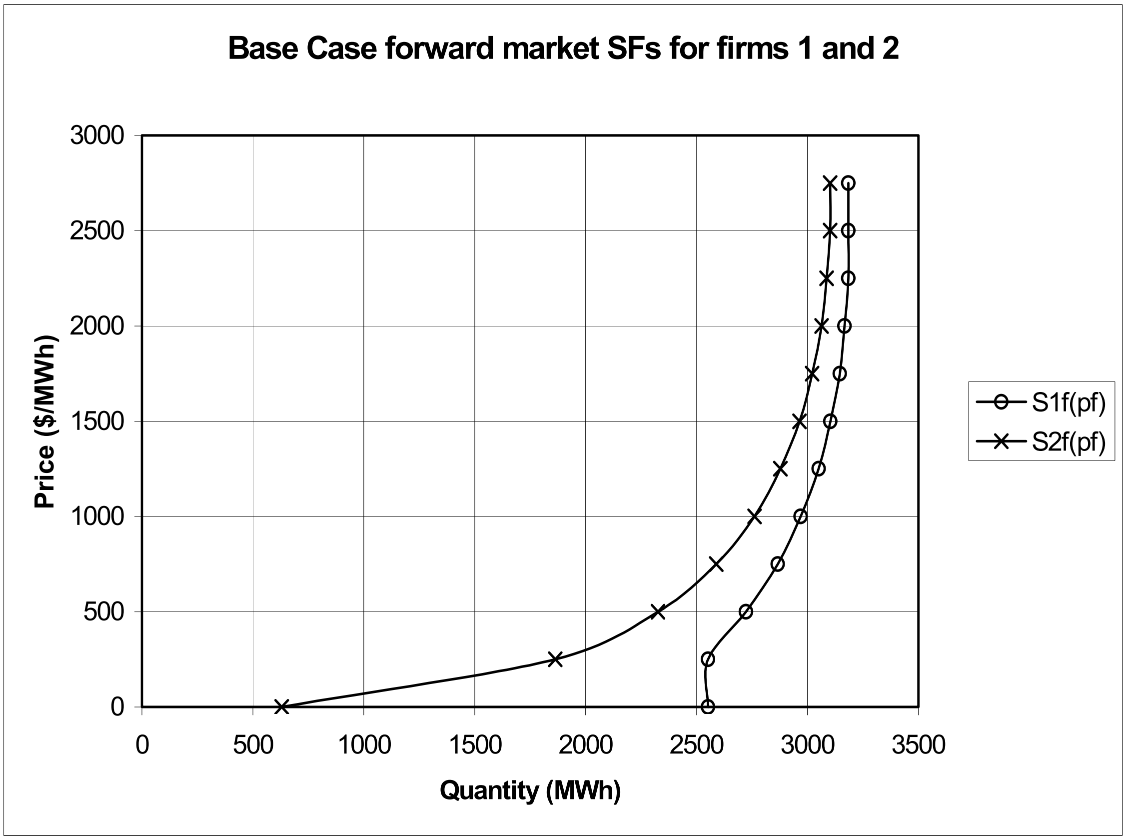

FIGURE 7.19: BASE CASE FORWARD MARKET SUPPLY FUNCTIONS FOR COMPARATIVE STATICS ANALYSIS

The SFs depicted in Figure 7.19 for each firm are everywhere strictly increasing[298] and moreover, yield positive forward market quantities over the range of prices p^f depicted there. That is, both suppliers take short forward market positions $\overline{q}_i^f > 0$ in the base

[298] Consistent with the constraint—implicit in problem (7.58)—that $\overline{S}_i^f\left(p^f\right)$ be strictly increasing. As Figure 7.19 suggests, this constraint is binding for each firm at the highest price levels.

case.[299] The strictly increasing SFs in Figure 7.19 correspond to an SF trajectory that lies entirely in *Region I* of the phase space's upper partition (see, e.g., Figure 7.15 for $p^f < \$1{,}830/\mathrm{MWh}$). Finally, consistent with intuition, Figure 7.19 shows that the low-cost firm, firm 1, is more aggressive in the forward market, bidding a larger forward market quantity at all prices p^f.

7.6.2 Computation of forward market SFs: Test case problems

Let θ represent an arbitrary element of Θ. We define the *test case* for the parameter θ as a solution of the multi-settlement SFE model (as approximated by the discrete Excel model) in which the parameter θ—and only that element—is perturbed from its value in the base case vector Θ^{base} (recall eq. (7.46)). We denote the resulting *test case vector* as the parameter vector Θ_θ^{test} for the perturbation of the parameter θ. The comparative statics analysis described below consists of perturbing each parameter θ in Θ from its value in Θ^{base} with a multiplicative shock of 1.001.[300] We did so one parameter at a time to obtain ten different test vectors Θ_θ^{test}.

The *test case problem* for parameter θ again relies on an optimization problem having the general form of problem (7.43). Beginning with problem (7.58), we replace Θ^{base} with Θ_θ^{test} to obtain a family of test case problems, one for each parameter θ:

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299 Allaz's (1992, 299ff.) observation is apropos, namely, that whether suppliers are short or long in the forward market is sensitive, in particular, to the type of conjectural variation as well as suppliers' attitudes toward risk.

300 That is, the parameter θ changes by 0.1% between the vectors Θ^{base} and Θ_θ^{test}. Through experimentation, we find that this small multiplicative shock is large enough to avoid spurious numerical results, but small enough to interpret the change in the parameter as a marginal change. See Appendix E.4 for further details.

$$\min_{\overline{S}_i^f\left(p^f\right)\ (discretized)} \left[\frac{\mathrm{E}\left(p^s\right)-p_{empir}^{s,\,mean}}{p_{empir}^{s,\,mean}}\right]^2+\left[\frac{\mathrm{E}\left(\overline{q}_{Agg}^s\right)-q_{empir}^{s,\,mean}}{q_{empir}^{s,\,mean}}\right]^2$$

$$\begin{aligned} s.t. \quad & \text{Subgame-perfect Nash equilibrium in } \overline{\Sigma}_i^s \text{ and } \overline{S}_i^f \\ & s.t. \quad \overline{q}_i^s \geq 0 \\ & \qquad\quad \text{Parameters } \Theta_\theta^{test}. \end{aligned} \tag{7.61}$$

The only distinction between the base case problem (7.58) and the test case problem (7.61) is the perturbation of the parameter θ, that is, the use of Θ^{base} versus Θ_θ^{test}.

Note that while the two spot market benchmarking constraints $\mathrm{E}\left(p^s\right)=p_{empir}^{s,\,mean}$ and $\mathrm{E}\left(\overline{q}_{Agg}^s\right)=q_{empir}^{s,\,mean}$ happen to hold in the base case problem (7.58), these constraints are not explicitly imposed, either in the base case problem (7.58) or the test case problem (7.61). In the solutions to the various test cases for arbitrary parameter perturbations, these constraints will not necessarily hold. For sufficiently small parameter perturbations, however, we would expect the objective function in the problem (7.61) to be close to zero; this is indeed the case.

7.6.3 Results and interpretation

Table 7.2 below reports the effects of perturbing each of the ten comparative statics parameters on firms' forward market quantities, that is, on the SFs $\overline{S}_1^f\left(p^f\right)$ and $\overline{S}_2^f\left(p^f\right)$ computed from problem (7.61).[301] Appendix E.4 reports the numerical results for the discretized SFs $\overline{S}_1^f\left(p^f\right)$ and $\overline{S}_2^f\left(p^f\right)$ in the base case and test cases. These results form the basis of the qualitative effects reported in Table 7.2. In the third and fourth columns

[301] We solve problem (7.61) without using automatic scaling in the Excel Solver (see note 284).

of the table, we use the symbols "+" and "–" to denote an increase and a decrease, respectively, in a firm's quantity attributable to the perturbation in question. Except in two cases, the direction that the SFs shift in response to a perturbation is monotone, that is, uniform across prices p^f in the chosen range of integration. The two exceptional cases in which the perturbation of one or both SFs is not monotone are (1) the change in $\overline{q}_1^f = \overline{S}_1^f\left(p^f\right)$ due to perturbation of c_{01}, and (2) the change in $\overline{q}_2^f = \overline{S}_2^f\left(p^f\right)$ due to perturbation of $\overline{v}_R$. The qualitative effects in both of these cases are an increase in the indicated firm's quantity at higher prices and a decrease at lower prices, which we indicate in Table 7.2 with the symbol "$\pm$."

In contrast, in each case examined, the *amount* that the SFs shift in response to parameter perturbations does depend on price, as is evident from the quantities in the columns of Table E.1 labeled "Δ" (see Appendix E.4). In general, therefore, these parameter variations change the slopes as well as higher derivatives of the SFs. The present discussion of comparative statics effects largely abstracts, however, from such higher-order changes in the SFs, focusing instead on the changes in quantities summarized in Table 7.2 below.

TABLE 7.2: COMPARATIVE STATICS ANALYSIS: EFFECTS OF PARAMETER PERTURBATIONS ON FIRMS' QUANTITIES SUPPLIED IN THE FORWARD MARKET, $\overline{q}_1^f = \overline{S}_1^f\left(p^f\right)$ AND $\overline{q}_2^f = \overline{S}_2^f\left(p^f\right)$

Parameter θ [a]	*Description* [b]	*Effect on* $\overline{q}_1^f = \overline{S}_1^f\left(p^f\right)$ [c]	*Effect on* $\overline{q}_2^f = \overline{S}_2^f\left(p^f\right)$ [c]
c_{01}	Price-axis intercept of firm 1's marginal cost function	±	+
c_{02}	Price-axis intercept of firm 2's marginal cost function	+	+
c_1	Slope of firm 1's marginal cost function	–	–
c_2	Slope of firm 2's marginal cost function	–	–
e_{dem}^s	Spot market demand elasticity	+	+
$\overline{\eta}_R$	Mean of representative consumer R's signal η_R	+	+
$\sigma_{\eta_R}^2$	Variance of representative consumer R's signal η_R	+	+
$\overline{\nu}_R$	Mean of spot market noise parameter ν_R	+	±
$\sigma_{\nu_R}^2$	Variance of spot market noise parameter ν_R	–	–
λ_R	Representative consumer R's parameter of constant absolute risk aversion (CARA)	–	–

Notes:

[a] See eq. (7.46) for the base case values Θ^{base} of each parameter θ.

[b] Recall from eq. (7.46) that firm 1 is a low-cost firm and firm 2 a high-cost firm.

[c] The symbols "+" and "–" denote an increase and a decrease, respectively, in a firm's quantity (at all prices p^f) attributable to the perturbation under study. The symbol "±" denotes an increase in the firm's quantity at higher prices p^f and a decrease in this quantity at lower prices.

In the bulleted paragraphs that follow, we provide intuition underlying the comparative statics effects documented in Table 7.2 above.[302] This discussion relies upon several properties of the multi-settlement SFE model and its solution. Among these properties are the elasticities of supply functions and demand functions in each market, the asymmetry of the two firms, endogeneity of forward market demand, and the risk preferences of the market participants. We appeal repeatedly to these features of the model in the following discussion.

- *An increase in firm 1's marginal cost function intercept c_{01} has a price-dependent effect on $\bar{S}_1^f\left(p^f\right)$. Namely, $\bar{S}_1^f\left(p^f\right)$ rotates clockwise, implying an increase in $\bar{q}_1^f$ at higher prices p^f, and a decrease in $\bar{q}_1^f$ at lower prices. The effect on $\bar{S}_2^f\left(p^f\right)$, in contrast, is monotone, shifting this forward market SF to the right.*

  Consider first firm 1. We begin by accounting for the rightward shift in $\bar{S}_1^f\left(p^f\right)$ at higher (indeed, most) values of p^f, and then consider why the direction of this shift may be reversed for sufficiently low p^f. From the geometry of the spot market examined in chapter 5, an increase in c_{01} will shift $\bar{\Sigma}_1^s$ upward, leading to higher equilibrium prices p^s and lower quantities $\bar{q}_1^s$ for every realization of ε^s. These changes due to increased c_{01} imply higher point elasticities of supply, of demand, and hence of residual demand. As a consequence, firm 1 can increase its

[302] Table E.1 of Appendix E.4 also includes numerical comparative statics results for the effects of parameter variations—namely, for the parameters c_i and γ^s—on SF slopes β_i^s. These comparative statics effects are among results reported in section 5.3 above for the spot market.

spot market quantity $\overline{q}_1^s$ with proportionally little penalty in terms of lower p^s. Under these circumstances, it tends to be profitable for firm 1 to increase its spot market quantity. One means by which it may do so is to increase its *forward market* quantity $\overline{q}_1^f$, since $\overline{q}_1^f$ shifts $\overline{\Sigma}_1^s$ to the right, in equilibrium. Firm 1 accomplishes this increase—at least for higher values of p^f—by shifting $\overline{S}_1^f\left(p^f\right)$ to the right.

Next, we examine why the direction of firm 1's incentive as sketched above might be reversed for sufficiently low p^f, causing instead a leftward shift in $\overline{S}_1^f\left(p^f\right)$ at such prices (and leading, in effect, to the clockwise rotation of $\overline{S}_1^f\left(p^f\right)$). The increase in c_{01} implies a uniform increase in firm 1's marginal cost, making it a less aggressive competitor, manifested in part by the aforementioned upward shift in $\overline{\Sigma}_1^s$. If, in addition, forward market demand is weak, then the equilibrium forward market price p^f will be low. This implies, in turn, that $\mathrm{E}\left(p^s\right)$ is also low. Under these circumstances, returns to firm 1 from decreasing $\overline{q}_1^f$ to support p^f could outweigh the prospects for increased expected spot market profits described in the foregoing paragraph. Accordingly, at sufficiently low values of p^f, firm 1 shifts its forward market SF to the left.[303] The net effect is the clockwise rotation of $\overline{S}_1^f\left(p^f\right)$.

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[303] Specifically, from the "c_{01}" scenario in Table E.1 of Appendix E.4, this is the case for $p^f \leq \$250/\mathrm{MWh}$.

Consider now the reaction of firm 2 to the increase in c_{01}. Firm 2 does not change its spot market SF $\overline{\Sigma}_2^s$ directly in response to firm 1's cost increase.[304] The upward shift in $\overline{\Sigma}_1^s$, however, shifts firm 2's spot market residual demand function RD_2^s upward, as well, increasing for each ε^s the equilibrium price p^s that firm 2 faces. In response, it is profitable for firm 2 to increase its equilibrium quantity $\overline{q}_2^s$. Firm 2 can do this (analogously to the argument above for firm 1) by increasing $\overline{q}_2^f$. The firm does so, in turn, by shifting its forward market SF $\overline{S}_2^f\left(p^f\right)$ to the right.

Now compare the relative responses of the two firms to the increase in c_{01}. Note from Table E.1 in Appendix E.4 that firm 1 increases its forward market quantity by a lesser amount at each price than does firm 2, so that the overall effect of increasing c_{01} is for firm 1 to cede some market share to firm 2 in *both* markets. Finally, we consider why increased c_{01} might cause firm 1 to decrease $\overline{S}_1^f\left(p^f\right)$ at low p^f, while firm 2's forward market SF $\overline{S}_2^f\left(p^f\right)$, in contrast, increases over the entire range of p^f considered here. One conjecture arises, naturally, from the asymmetry in firms' marginal cost functions. Since firm 2 is the high cost firm, we have that $\overline{\Sigma}_2^s$ is steeper than $\overline{\Sigma}_1^s$. This differential in the slopes of the spot market SFs means that the slopes of firms' spot market residual

[304] Though $\overline{\Sigma}_2^s$ does shift to the right with the increase in $\overline{q}_2^f$ that we describe below, in accordance with the analysis of chapter 5.

demand functions RD_i^s have the opposite relationship. That is, RD_1^s is steeper than RD_2^s, and thus the magnitude of firm 2's residual demand elasticity tends to be greater than that for firm 1. The relatively inelastic function RD_1^s implies that firm 1 is more likely than firm 2 to profit from decreasing its spot market quantity, thereby driving up the equilibrium price p^s. In some states of the world—namely, at low p^f, as argued above, where forward market marginal revenues are relatively low—it is profitable for firm 1 to do just this by decreasing $\overline{q}_1^f = \overline{S}_1^f\left(p^f\right)$, thus shifting $\overline{\Sigma}_1^s$ to the left.

- *An increase in firm 2's marginal cost function intercept c_{02} shifts both firms' forward market SFs to the right.*

  For each firm, the argument here is analogous to that for firm 2 in the above discussion regarding the effects of increased c_{01}. Other things equal, the increase in c_{02} increases p^s. Both firms have an incentive to profit from increased spot market prices by increasing equilibrium quantities $\overline{q}_i^s$. Each firm can do this by increasing its forward market quantity $\overline{q}_i^f$. The firms do so, in turn, by shifting their forward market SFs $\overline{S}_i^f\left(p^f\right)$ to the right.

Similar to the argument for c_{01} above, Table E.1 in Appendix E.4 indicates that firm 2 increases its forward market quantity by a lesser amount at each price than does firm 1. The overall effect of increasing c_{02}, therefore, is for firm 2 to cede some market share to firm 1 in both markets.

- *An increase in firm 1's marginal cost function slope c_1 shifts both firms' forward market SFs to the left.*

  The effects of the increase in c_1 include decreases in both β_1^s and β_2^s, that is, steeper spot market SFs $\overline{\Sigma}_i^s$. Steeper SFs $\overline{\Sigma}_i^s$—in effect, a counter-clockwise rotation of these functions $\overline{\Sigma}_i^s$—are less elastic, and lead also to less elastic spot market residual demand functions. Such low elasticities in the spot market tend to make increases in the expected spot market price $\mathrm{E}\left(p^s\right)$ profitable.[305] In this scenario, less elastic SFs $\overline{\Sigma}_i^s$ imply that it would be profitable for firms to decrease their forward market quantities via a leftward shift in $\overline{S}_i^f\left(p^f\right)$. This is because even only a small decrease in $\overline{q}_i^f$ will drive $\mathrm{E}\left(p^s\right)$ markedly higher, with little change in $\overline{q}_i^s$ and relatively little sacrifice in forward market revenue.

- *An increase in firm 2's marginal cost function slope c_2 shifts both firms' forward market SFs to the left.*

  The effects of the increase in c_2 are analogous to those for increased c_1, discussed above. That is, less elastic SFs $\overline{\Sigma}_i^s$ imply that it would be profitable for firms to decrease their forward market quantities via a leftward shift in $\overline{S}_i^f\left(p^f\right)$.

[305] Note that with the rotation of the functions $\overline{\Sigma}_i^s$, the size of the price increase for a given increment in c_1 increases with the initial value of $\mathrm{E}\left(p^s\right)$.

- *An increase in the magnitude of the spot market demand elasticity e_{dem}^{s} shifts both firms' forward market SFs to the right.*

  As the spot market demand elasticity e_{dem}^{s} increases in magnitude, both firms will face a lower penalty in the expected spot market price $\mathrm{E}\left(p^{s}\right)$ from expansion of their respective spot market outputs. This change increases the elasticity of each firm's spot market residual demand function, implying that greater expected spot market quantities are now profitable. Given that, for each firm, $\overline{q}_{i}^{f}$ shifts $\overline{\Sigma}_{i}^{s}$ to the right, it is optimal for firms to increase $\overline{q}_{i}^{f}$ in response to the increase in e_{dem}^{s}. This implies that $\overline{S}_{i}^{f}\left(p^{f}\right)$ shifts to the right.

- *An increase in the mean $\overline{\eta}_{R}$ of the representative consumer R's signal shifts both firms' forward market SFs to the right.*

  The parameter $\overline{\eta}_{R}$ does not appear in the firms' forward market equilibrium optimality conditions, eqs. (7.11) and (7.12). As a consequence, a shock to $\overline{\eta}_{R}$ while holding constant the initial conditions for the SFs $\overline{S}_{i}^{f}\left(p^{f}\right)$ leaves these SFs unaffected.[306] Note, however, that an increase in $\overline{\eta}_{R}$ does affect the *unconditional* expectations of spot market price $\mathrm{E}\left(p^{s}\right)$ and quantity $\mathrm{E}\left(\overline{q}_{Agg}^{s}\right)$ in the objective functions of the base case problem (7.58) and the test case problem (7.61). In this way, the equilibrium selection algorithm in these problems depends on η_{R}'s

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[306] This must be the case given that we derived the optimality conditions for the forward market problem conditional on p^{f} *and* η_{R}. Conditioning on the realization η_{R} renders the solution invariant to changes in η_{R}'s distribution.

distribution; in particular, the equilibrium selected by this problem varies with $\overline{\eta}_R$. The important general result here is that—apart from equilibrium selection considerations—the forward market SFs are *independent* of the distribution of the signal η_R.

To understand the intuition behind the rightward shift in the SFs observed for an increase in $\overline{\eta}_R$, begin by recalling the simple additive relationship $\overline{\varepsilon}^s = \overline{\eta}_R + \overline{v}_R$ among the means of the stochastic parameters (see eq. (6.55)). From this equality, an increase in $\overline{\eta}_R$ (holding constant, for the moment, the forward and spot market SFs) increases both the expected spot market price $\mathrm{E}\left(p^s\right)$ and the expected aggregate spot market quantity $\mathrm{E}\left(\overline{q}^s_{Agg}\right)$. Due, however, to the inelastic spot market demand function $D^s\left(p^s, \varepsilon^s\right)$, the proportional change[307] in $\mathrm{E}\left(p^s\right)$ is much greater than that in $\mathrm{E}\left(\overline{q}^s_{Agg}\right)$, which remains approximately constant (and can hence be neglected in this discussion). If we now solve the test case problem (7.61) given the increase in $\overline{\eta}_R$, the SFs $\overline{S}_i^f\left(p^f\right)$ change so as to minimize the objective function of this problem. To minimize this function, $\mathrm{E}\left(p^s\right)$ must decrease to offset the increase in $\mathrm{E}\left(p^s\right)$ which would otherwise occur, as noted above. To effect this decrease in $\mathrm{E}\left(p^s\right)$, forward market quantities must increase, corresponding to rightward shifts in the SFs $\overline{S}_i^f\left(p^f\right)$.

[307] It is the proportional change in $\mathrm{E}\left(p^s\right)$ or $\mathrm{E}\left(\overline{q}^s_{Agg}\right)$ that is relevant for the objective functions of problems (7.58) and (7.61).

- *An increase in the variance $\sigma^2_{\eta_R}$ of the representative consumer R's signal shifts both firms' forward market SFs to the right.*

  The intuition in this case is very similar to that in the preceding case investigating the effects of an increase in $\bar{\eta}_R$. That is, like $\bar{\eta}_R$, the parameter $\sigma^2_{\eta_R}$ does not appear in the firms' forward market equilibrium optimality conditions, eqs. (7.11) and (7.12). As a result, a shock to $\sigma^2_{\eta_R}$ while holding constant the initial conditions for the SFs $\bar{S}_i^f\left(p^f\right)$ leaves these SFs unaffected. As with $\bar{\eta}_R$, the increase in $\sigma^2_{\eta_R}$ does affect the unconditional expectations of spot market price $\mathrm{E}\left(p^s\right)$ and quantity $\mathrm{E}\left(\bar{q}^s_{Agg}\right)$ in the objective functions of the base case problem (7.58) and the test case problem (7.61), so that the equilibrium selected by these problems varies with $\sigma^2_{\eta_R}$.

  We may show numerically that an increase in $\sigma^2_{\eta_R}$ (holding constant, at first, the forward and spot market SFs) increases the unconditional expected spot market price $\mathrm{E}\left(p^s\right)$ and decreases the unconditional expected spot market quantity $\mathrm{E}\left(\bar{q}^s_{Agg}\right)$. Due to the inelastic spot market demand function $D^s\left(p^s,\varepsilon^s\right)$, the proportional change in $\mathrm{E}\left(p^s\right)$ is again much greater than that in $\mathrm{E}\left(\bar{q}^s_{Agg}\right)$, which remains approximately constant (and can again be neglected in this discussion). Solving the test case problem (7.61) given the increase in $\sigma^2_{\eta_R}$, the SFs $\bar{S}_i^f\left(p^f\right)$ change so as to minimize the objective function of this problem. To minimize this function, $\mathrm{E}\left(p^s\right)$ must decrease to offset the increase in $\mathrm{E}\left(p^s\right)$

which would otherwise occur, as noted above. To effect this decrease in $\mathrm{E}\left(p^s\right)$, forward market quantities must increase, corresponding to rightward shifts in the SFs $\bar{S}_i^f\left(p^f\right)$.

- *An increase in the mean $\bar{\nu}_R$ of the spot market noise parameter has a monotone effect on $\bar{S}_1^f\left(p^f\right)$, shifting this forward market SF to the right. The effect on $\bar{S}_2^f\left(p^f\right)$, in contrast, is price-dependent. Namely, $\bar{S}_2^f\left(p^f\right)$ rotates clockwise, implying an increase in $\bar{q}_2^f$ at higher prices p^f, and a decrease in $\bar{q}_2^f$ at lower prices.*

Below, we first explain the effect—predominant for both firms, at most prices—of increasing $\bar{q}_i^f = \bar{S}_i^f\left(p^f\right)$, and then address the question of firm 2's distinct forward market behavior at low p^f.

Since $\bar{\varepsilon}^s = \bar{\eta}_R + \bar{\nu}_R$, the increase in $\bar{\nu}_R$ increases expected spot market demand, shifting the function $\mathrm{E}\left(D^s\left(p^s, \varepsilon^s\right)\right)$ to the right. Moreover, we may show numerically that the increase in $\bar{\nu}_R$ shifts the expected forward market demand function $\mathrm{E}\left(D^f\left(p^f, \varepsilon_0^f\right)\right)$ to the left at higher (and indeed, most) prices p^f, and to the right at sufficiently low p^f (i.e., at $p^f \leq \$250/\mathrm{MWh}$), effectively rotating $\mathrm{E}\left(D^f\left(p^f, \varepsilon_0^f\right)\right)$ counterclockwise. These changes in $\mathrm{E}\left(D^f\left(p^f, \varepsilon_0^f\right)\right)$ make this function more elastic and decrease the expected forward market price $\mathrm{E}\left(p^f\right)$. Consistent with these changes, the $\bar{S}_i^f\left(p^f\right)$ also become more elastic.

Moreover, for any realization of ε_0^f, the functions $D^f\left(p^f, \varepsilon_0^f\right)$ and $\overline{S}_i^f\left(p^f\right)$ are negatively related (ceteris paribus), and thus the $\overline{S}_i^f\left(p^f\right)$ tend to shift to the right, in opposition to the shift in $D^f\left(p^f, \varepsilon_0^f\right)$. The aforementioned shifts in $\overline{S}_i^f\left(p^f\right)$ and $D^f\left(p^f, \varepsilon_0^f\right)$ imply that the forward market equilibrium moves toward the elastic range of both of these functions (i.e., to lower values of the equilibrium forward market price p^f). As a consequence, suppliers may increase their forward market quantities $\overline{q}_i^f$ with little downward pressure on p^f. Forward market revenue generally increases with such an increase in $\overline{q}_i^f$, and thus a rightward shift in $\overline{S}_i^f\left(p^f\right)$ tends to be profitable for each supplier.

Consider now firm 2's distinct reaction at low forward market prices p^f to increased $\overline{v}_R$. Recall that firms 1 and 2 have asymmetric cost functions. As the higher cost firm, firm 2 is a less aggressive competitor than is firm 1. To be profitable, firm 2 requires a higher equilibrium price (in either market) than does firm 1. When firm 1 puts downward pressure on an already low forward market price p^f by increasing $\overline{q}_1^f$, firm 2's optimal response may be to decrease $\overline{q}_2^f$ to support p^f (and also p^s). In some states of the world—namely, at low p^f, where firms' forward market marginal revenues are relatively low—it is profitable for firm 2 to respond in exactly this way by shifting $\overline{S}_2^f\left(p^f\right)$ to the left.

Finally, we compare the relative responses of the two firms to the increase in $\overline{v}_R$. Table E.1 in Appendix E.4 indicates that firm 1 increases its forward market quantity by a greater amount at each price than does firm 2. Accordingly, the overall effect on forward market competition of increasing $\bar{v}_R$ is for firm 2 to cede some market share to firm 1 in the forward market.

- *An increase in the variance $\sigma^2_{v_R}$ of the spot market noise parameter shifts both firms' forward market SFs to the left.*

The two suppliers are risk neutral, and care only about an increase in $\sigma^2_{v_R}$ through its effect on demand and on expected spot market prices (see eqs. (7.8) and (7.9)). A risk-averse consumer, on the other hand, does respond directly to the change in $\sigma^2_{v_R}$, and a change in the forward market demand function will affect the simultaneously-determined SFs $\bar{S}_i^f\left(p^f\right)$.

By the above reasoning, it is useful to begin by considering the effect of $\sigma^2_{v_R}$ on forward market demand $D^f\left(p^f,\varepsilon_0^f\right)$. We can show numerically at base case parameter values that, for a given realization of ε_0^f, increasing $\sigma^2_{v_R}$ shifts $D^f\left(p^f,\varepsilon_0^f\right)$ to the right and makes this demand function less elastic at all prices. The intuition underlying these effects is as follows. Since $\sigma^2_{\eta_R}$ is held constant in this scenario, the posited increase in $\sigma^2_{v_R}$ increases the relative risk of the spot market. This change, in turn, leads a risk-averse consumer, ceteris paribus, to reduce her exposure to the spot market price. Accordingly, the consumer then demands higher forward market quantities, and forward market demand becomes less price-sensitive.

Consistent with these changes, the $\overline{S}_i^f\left(p^f\right)$ also become less elastic.[308] Moreover, because $D^f\left(p^f,\varepsilon_0^f\right)$ and $\overline{S}_i^f\left(p^f\right)$ are negatively related (ceteris paribus), the $\overline{S}_i^f\left(p^f\right)$ shift to the left. The aforementioned shifts in $\overline{S}_i^f\left(p^f\right)$ and $D^f\left(p^f,\varepsilon_0^f\right)$ imply that the forward market equilibrium moves toward the inelastic range of both of these functions, and the equilibrium forward market price p^f is driven up. When both $\overline{S}_i^f\left(p^f\right)$ and $D^f\left(p^f,\varepsilon_0^f\right)$ are inelastic, decreasing $\overline{S}_i^f\left(p^f\right)$ increases markedly the equilibrium forward market price p^f. This change in $\overline{S}_i^f\left(p^f\right)$ increases firms' forward market revenues, and hence the leftward shift in $\overline{S}_i^f\left(p^f\right)$ is profitable for each supplier.

- *An increase in the representative consumer R's CARA parameter λ_R shifts both firms' forward market SFs to the left.*

  As in the above analysis for shocks to the parameters $\overline{v}_R$ and $\sigma_{v_R}^2$, it is useful to begin analysis of this scenario by considering the effect of λ_R on forward market demand $D^f\left(p^f,\varepsilon_0^f\right)$. Increased λ_R implies that the representative consumer *R* is more sensitive to risk. Since risk in this problem may be proxied by $\sigma_{v_R}^2$, the effect of increased *sensitivity to* $\sigma_{v_R}^2$ is qualitatively equivalent to the effect of increased $\sigma_{v_R}^2$ (with constant λ_R), analyzed above.

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[308] Except perhaps at the highest prices p^f.

Accordingly, we can show numerically at base case parameter values that, for a given realization of ε_0^f, increasing λ_R shifts $D^f\left(p^f,\varepsilon_0^f\right)$ to the right and makes this demand function less elastic at all prices. The intuition here is that an increasingly risk-averse consumer demands higher forward market quantities, and that forward market demand becomes less price-sensitive as consumers' risk aversion increases. Consistent with these changes, because $D^f\left(p^f,\varepsilon_0^f\right)$ and $\overline{S}_i^f\left(p^f\right)$ are negatively related (ceteris paribus), the $\overline{S}_i^f\left(p^f\right)$ shift to the left. The aforementioned shifts in $\overline{S}_i^f\left(p^f\right)$ and $D^f\left(p^f,\varepsilon_0^f\right)$ imply that the forward market equilibrium moves toward the inelastic range of both of these functions, and the equilibrium forward market price p^f is driven up. When both $\overline{S}_i^f\left(p^f\right)$ and $D^f\left(p^f,\varepsilon_0^f\right)$ are inelastic, decreasing $\overline{S}_i^f\left(p^f\right)$ increases markedly the equilibrium forward market price p^f. This change in $\overline{S}_i^f\left(p^f\right)$ increases firms' forward market revenues, and hence the leftward shift in $\overline{S}_i^f\left(p^f\right)$ is profitable for each supplier.

As a unifying framework for understanding the comparative statics results documented in this subsection, we may focus on the effect of parameter shocks on the *elasticity of residual demand functions* in each market at the respective equilibrium points. From the qualitative analysis of this subsection, we may conclude that the elasticity of spot market residual demand increases for increases in c_{0i} and e_{dem}^s, while this elasticity decreases for increases in c_i. Similarly, the elasticity of forward market residual demand generally increases for increases in $\bar{\eta}_R$, $\sigma^2_{\eta_R}$, and $\bar{\nu}_R$, while this elasticity decreases for increases in $\sigma^2_{\nu_R}$ and λ_R. Parameter changes that *increase* the elasticity of residual demand in either the forward or spot markets tend, in general, to make firms more aggressive in the forward market in that they bid higher quantities at each price. That is, rightward shifts in $\bar{S}_i^f\left(p^f\right)$ are the result of such changes. The converse is true for parameter changes that *decrease* the elasticity of residual demand in either market. In other words, such changes cause leftward shifts in $\bar{S}_i^f\left(p^f\right)$. This behavior is consistent with intuition regarding a profit-maximizing firm's best responses to such shocks.

7.7 Comparison of expected aggregate welfare under alternative behavioral assumptions and market architectures

We conclude this chapter by comparing expected aggregate welfare for the multi-settlement SFE model with that obtained from models employing alternative behavioral assumptions and market architectures. In particular, we compare the multi-settlement SFE model to two alternative single-market models:[309]

1. *Single-market SFE*: We assume away the forward market, and assume further (as in the multi-settlement SFE model) that firms bid affine SFs in the spot market.[310]

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[309] Like the multi-settlement SFE model, both alternative models assume duopoly suppliers.

[310] This is the scenario that Klemperer and Meyer (1989) examine.

2. *Perfect competition (single market)*: Again, we assume away the forward market, and moreover, assume that firms behave competitively, bidding their marginal cost functions in place of supply functions in the spot market.

To compute a welfare measure for the multi-settlement SFE model, we assume a risk-neutral social planner who assesses welfare ex ante under uncertainty using the mathematical expectation of a utilitarian social welfare function. In the partial equilibrium framework invoked here, only electricity is produced and consumed (apart from the numeraire good *m*). Therefore, in either the multi-settlement SFE model or the two alternative models noted above, *expected aggregate welfare* $\mathrm{E}\left(W_{Agg}\right)$ consists of two components:

1. the expected utility $\mathrm{E}\left[\phi\left(x_R\right)\right]$ of the representative consumer *R*'s consumption of amenity x_R (produced using electricity as an input, recalling eqs. (6.2) and (6.1)) and
2. the expected total cost of production $\mathrm{E}\left[C\left(q_R^s\right)\right]$ of the equilibrium quantity of electricity q_R^s used by *R*, where

$$\mathrm{E}\left[C\left(q_R^s\right)\right]=\mathrm{E}\left[\sum_{i=1}^{2} C_i\left(\overline{q}_i^s\right)\right]=\mathrm{E}\left[\sum_{i=1}^{2}\left(\int_0^{\overline{q}_i^s} C_i'\left(q_i^s\right)dq_i^s\right)\right]$$

and eq. (5.1) gives each firm *i*'s affine marginal cost function $C_i'\left(q_i^s\right)$. Algebraically, $\mathrm{E}\left(W_{Agg}\right)$ is the difference of the utility and total cost terms above, that is,[311]

[311] In eq. (7.62), we compute expectations with respect to both spot and forward market sources of uncertainty via the discrete Excel model (see subsection 7.3.2).

$$\mathrm{E}\left(W_{Agg}\right)=\mathrm{E}\left[\phi\left(x_R\right)\right]-\mathrm{E}\left[\sum_{i=1}^{2}\left(\int_0^{\overline{q}_i^s} C_i'\left(q_i^s\right)dq_i^s\right)\right]. \tag{7.62}$$

By definition, expected aggregate welfare in eq. (7.62) does *not* account for transfers—due, in particular, to forward market activity—between consumers and producers. If such distributional effects are also of interest to policy makers, it is straightforward, for example, to use the present model to compute moments of the distribution of consumers' forward market payments to suppliers.

In Table 7.3 below, we use eq. (7.62) to compute expected aggregate welfare $\mathrm{E}\left(W_{Agg}\right)$ for the multi-settlement SFE model and the two alternative models noted above.

TABLE 7.3: EXPECTED AGGREGATE WELFARE $\mathrm{E}\left(W_{Agg}\right)$ FOR THE MULTI-SETTLEMENT SFE MODEL AND ALTERNATIVE MODELS

Model[a]	*Expected aggregate welfare ($)*[b]	*Percentage of expected aggregate welfare in perfectly competitive model*
Multi-settlement SFE	302,265.90	94.75%
Single-market SFE	294,505.97	92.32%
Perfect competition (single market)	319,003.66	100.00%

Notes:

[a] Each model assumes base case parameters Θ^{base} from eq. (7.46).

[b] We compute expected aggregate welfare assuming the following values for the parameters of the representative consumer *R*'s amenity production function $f\left(q_R^s, T_R\right)$ and utility function $\phi\left(x_R\right)$: $a_0 = 2$, $a_1 = 40$, $a_2 = 0.4$, and $b \approx 225$ (recall that b is endogenous to the slope γ^s of the spot market demand function; see section 6.4 for details). The relative welfare ranking of the various models does not change, however, for alternative choices of these parameters.

As intuition would suggest, Table 7.3 indicates that the perfectly competitive scenario has the highest value of expected aggregate welfare. The multi-settlement SFE model has the next highest figure for expected aggregate welfare, and the single-market SFE the smallest. Comparing the multi-settlement SFE model with the single-market

SFE model, we see that—as we would expect for a one-shot equilibrium analysis—introducing a forward market has a welfare-enhancing effect.[312] Namely, expected aggregate welfare for the multi-settlement SFE model exceeds that for the single-market SFE model by \$7759.93 (2.63%). Recall from eq. (7.46) for Θ^{base} that the spot market demand function underlying Table 7.3's scenarios is nearly perfectly inelastic (i.e., $e^{s}_{dem} = -5.95\text{e-}5$). If this function were more elastic, then the deviation of expected aggregate welfare between each SFE scenario in Table 7.3, on the one hand, and perfect competition, on the other, would be greater. Finally, we note that the welfare-enhancing property of forward markets is consistent with previous literature on multi-settlement markets reviewed in chapter 1, in particular, Allaz (1987), Allaz and Vila (1993), Powell (1993), Green (1999a), and Kamat and Oren (2002).

[312] This conclusion does not necessarily apply in a repeated game setting.

> [M]onopoly, in all its forms, is the taxation of the industrious for the support of indolence, if not of plunder.
>
> —John Stuart Mill, *Principles of Economy*

> [I]ndustries differ one from the other, and the optimal mix of institutional arrangements for any one of them cannot be decided on the basis of ideology alone. The "central institutional issue of public utility regulation" remains . . . finding the best possible mix of inevitably imperfect regulation and inevitably imperfect competition.
>
> —Alfred E. Kahn, *The Economics of Regulation*

8 Discussion, conclusions, and further research

THIS CHAPTER begins in section 8.1 below by examining market participants' motives for forward market activity in the multi-settlement SFE model. Next, section 8.2 highlights potential avenues for future research by offering some preliminary conjectures on the implications of relaxing various model restrictions. Section 8.3 concludes the chapter by outlining how the results of the multi-settlement SFE model might be extended in further research to contribute to a framework for market power analysis.

8.1 Motives for forward market activity

Recalling from subsection 1.5.2 Allaz's (1987, 18) taxonomy of *hedging*, *speculative*, and *strategic motives* for forward market activity, this subsection examines which of these effects are present in the multi-settlement SFE model. Hedging and speculative motives are relatively transparent in this model, and hence easy to identify. We devote most of this section, accordingly, to analyzing strategic motives for forward market participation by the duopoly suppliers in the multi-settlement SFE model.

Because the suppliers are risk neutral in the present model, they do not exhibit hedging motives.[313] In contrast, speculative motives for suppliers to participate in the forward market do exist, since the conditional expectation of the forward contract cash flow $CF_i = \left(p^f - p^s\right)q_i^f$ enters firm i's profit maximization problem (eqs. (3.39) –(3.41)).[314] Finally, strategic motives are present for suppliers, as we explain below.

To motivate the discussion of strategic motives for suppliers' forward market activity, consider Green's (1999a, 115) observations that "a risk-neutral firm will not want to use a [forward] contract market unless this will affect its rival's strategy. By selling forward, a firm can increase its equilibrium output, but it will also reduce the price, just as if it had adopted a more aggressive strategy in the spot market. Since the firm could have adopted such a spot market strategy regardless of its position in the [forward] contract market, there has to be another mechanism at work to make selling contracts attractive. The opportunity to affect its rival's strategy is just such a mechanism." The multi-settlement SFE model, however, more closely resembles a

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[313] We could instead assume that suppliers are risk averse. It would then be appropriate for suppliers to maximize a utility function (e.g., of the mean-variance type) rather than simply to maximize profits. This change in objective function would produce hedging motives for suppliers. The ultimate effect on suppliers' forward market participation would then likely depend on the relative degree of uncertainty in the forward and spot markets.

[314] Subsection 7.6.1 noted that $\overline{q}_i^f > 0$ for base case parameter values over the range of forward market prices of interest. We conclude that the conditional expectation $\mathrm{E}\left[CF_i \middle| p^f\right]$ $= \left[p^f - \mathrm{E}\left(p^s \middle| p^f\right)\right]\overline{q}_i^f > 0$ since $\left[p^f - \mathrm{E}\left(p^s \middle| p^f\right)\right] > 0$ from inequality (7.51).

variant of Green's main model which he develops in an appendix to his 1999 paper (Green 1999b). In Green's alternative model, buyers are risk averse, which leads them in equilibrium to pay a hedging premium to suppliers in the forward market. As a consequence, the forward price exceeds the expected spot price. Under these circumstances, Green (1999b, 4) concludes that "[t]he ability to earn a hedging premium gives another motive for selling contracts, so that a firm will now hedge part of its output, even if this does not affect its rival's strategy and reduces its spot market profits, in order to earn a hedging premium." Due to the presence of risk-averse consumers and strategic suppliers in the multi-settlement SFE model, we find a set of incentives analogous to those in Green's alternative model with risk-averse buyers. Namely, we find that a risk-neutral firm has an incentive to participate in the forward contract market, in part to earn a hedging premium, and also to affect its rival's spot market stage game action. In the following subsections, we explore how this latter strategic motive for suppliers' forward market activity arises in the multi-settlement SFE model.

8.1.1 Effects of a supplier's forward market activity on equilibrium quantities

Given an arbitrary SF for firm 2, $S_2^f\left(p^f\right)$, let firm 1's best response to $S_2^f\left(p^f\right)$ be $S_1^f\left(p^f\right)$.[315] For a shock $\delta_1 > 0$, define from $S_1^f\left(p^f\right)$ a "base" forward market SF $\underline{S}_1^f\left(p^f\right) \equiv S_1^f\left(p^f\right) - \delta_1$ for firm 1, so that

$$S_1^f\left(p^f\right) = \underline{S}_1^f\left(p^f\right) + \delta_1 . \tag{8.1}$$

[315] Note that we refer here to an *optimal*—though not necessarily *equilibrium*—SF for firm 1, and hence use the notation $S_1^f\left(p^f\right)$ and q_1^f without the overbars "$^{-}$" that denote equilibrium functions and quantities.

Given the decomposition of $S_1^f\left(p^f\right)$ in eq. (8.1), we examine the effects of a differential shock $d\delta_1$ to firm 1's forward market SF bid, translating $S_1^f\left(p^f\right)$ to the right. Consistent with the development of the firm's forward market optimization problem in chapter 4, assume further that firm 1 imputes to firm 2 a fixed (disequilibrium) strategy of $\left\{S_2^f\left(p^f\right), \Sigma_2^s\left(p^s;q_2^f,q_1^f\right)\right\}$. Consider first the effects of the shift $d\delta_1$ in $S_1^f\left(p^f\right)$ on *forward market* competition. Obviously, this change increases firm 1's forward market quantity q_1^f at each price p^f. The rightward shift in $S_1^f\left(p^f\right)$ also causes firm 2's forward market residual demand function, $RD_2^f\left(p^f,\varepsilon_0^f\right) \equiv D^f\left(p^f,\varepsilon_0^f\right) - S_1^f\left(p^f\right)$, to shift to the left for fixed ε_0^f.[316] The function $RD_2^f\left(p^f,\varepsilon_0^f\right)$, of course, is the set of price-quantity tradeoffs that firm 2 faces in the forward market. For the present, suppose that $S_2^f\left(p^f\right)$ does not change. For fixed $S_2^f\left(p^f\right)$ and a leftward shift in $RD_2^f\left(p^f,\varepsilon_0^f\right)$, the quantity-price pair $\left(q_2^f, p^f\right)$ facing firm 2 moves downward and to the left. Accordingly, firm 2's forward market quantity q_2^f decreases, while the forward market-clearing price $p^f = p^{f*}\left(\varepsilon_0^f\right)$ decreases, as well.

Next, we examine the effects of the rightward shift in $S_1^f\left(p^f\right)$ on *spot market* competition. As shown in chapter 5 (see eq. (5.13) and Figure 5.1), this shift in $S_1^f\left(p^f\right)$

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[316] From the forward market analysis of chapter 7, the rightward shift in $S_1^f\left(p^f\right)$ affects $RD_2^f\left(p^f,\varepsilon_0^f\right)$ both directly, and through $D^f\left(p^f,\varepsilon_0^f\right)$. Since $D^f\left(p^f,\varepsilon_0^f\right)$ is endogenous (and negatively related) to the SFs $S_i^f\left(p^f\right)$ (recall eqs. (6.76) and (6.78)), the rightward shift in $S_1^f\left(p^f\right)$ shifts $D^f\left(p^f,\varepsilon_0^f\right)$, and hence $RD_2^f\left(p^f,\varepsilon_0^f\right)$, to the left.

translates, in turn, firm 1's spot market SF $\Sigma_1^s\left(p^s;q_1^f,q_2^f\right)$ to the right, increasing the firm's spot market quantity q_1^s for each p^s. This change in $\Sigma_1^s\left(p^s;q_1^f,q_2^f\right)$ implies further that firm 2's spot market residual demand function, $RD_2^s\left(p^s,\varepsilon^s;q_2^f,q_1^f\right)$ $\equiv D^s\left(p^s,\varepsilon^s\right)-\Sigma_1^s\left(p^s;q_1^f,q_2^f\right)$, shifts to the left for fixed ε^s (whereby $D^s\left(p^s,\varepsilon^s\right)$ is fixed). The function $RD_2^s\left(p^s,\varepsilon^s;q_2^f,q_1^f\right)$, of course, is the set of price-quantity tradeoffs that firm 2 faces in the spot market. The aforementioned decrease in q_2^f also shifts firm 2's spot market SF $\Sigma_2^s\left(p^s;q_2^f,q_1^f\right)$ to the left (see eq. (5.13)).[317] Since both $\Sigma_2^s\left(p^s;q_2^f,q_1^f\right)$ and $RD_2^s\left(p^s,\varepsilon^s;q_2^f,q_1^f\right)$ shift to the left, firm 2's spot market quantity q_2^s decreases (as we may confirm from the analysis of chapter 5). That is, we have that

$$\left.\frac{dq_2^s}{d\delta_1}\right|_{S_2^f(\bullet)\ \text{fixed}} < 0\,. \tag{8.2}$$

The sign of the net effect of leftward shifts in $\Sigma_2^s\left(p^s;q_2^f,q_1^f\right)$ and $RD_2^s\left(p^s,\varepsilon^s;q_2^f,q_1^f\right)$ on the spot market-clearing price p^s, however, is ambiguous in the general case.[318] In contrast, the sign of the effect of the rightward shift in $S_1^f\left(p^f\right)$ on firm 1's spot market quantity q_1^s is well-defined. Even if p^s should decrease, thereby putting downward

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[317] This is the effect of firm 1's forward market action (the change in which is the increment $d\delta_1$) on firm 2's spot market action, $\Sigma_2^s\left(p^s;q_2^f,q_1^f\right)$. See the quotation from Green (1999a, 115) beginning on page 327.

[318] In particular, this effect on p^s depends on the relative magnitudes of ϕ_1 and ϕ_2 as well as on the particular forward market SFs selected.

pressure on q_1^s, we may show from the analysis of chapter 5 that the net effect of increased q_1^f on q_1^s is positive, that is,[319]

$$\left.\frac{dq_1^s}{d\delta_1}\right|_{S_2^f(\cdot)\text{ fixed}} > 0 . \tag{8.3}$$

Of course, firm 2's forward market SF $S_2^f\left(p^f\right)$ need not—and in general will not—remain fixed (as in firm 1's imputation above) in response to $d\delta_1$. Rather, firm 2 chooses its SF (given the new SF for firm 1 $S_1^f\left(p^f\right)+d\delta_1$) according to the forward market optimization problem detailed in chapter 4 and sketched for firm 1 briefly above. In doing so, firm 2 faces a completely analogous set of incentives as those described previously for firm 1. Without repeating chapter 4's analysis, we next consider the likely nature of firm 2's best response to the increment $d\delta_1$ in firm 1's forward market SF posited above.

The preceding discussion indicated that an increment $d\delta_1$ caused both q_2^f and q_2^s to decrease, while the corresponding quantities for firm 1 increased. In other words, with a fixed $S_2^f\left(p^f\right)$, firm 2 would lose market share in both markets. As noted above, the effect of $d\delta_1$ on the spot market-clearing price p^s (with $S_2^f\left(p^f\right)$ fixed) was ambiguous. We cannot be precise about firm 2's best response to $d\delta_1$ without (1) specifying more

---

[319] To show this, differentiate eq. (5.9) totally with respect to δ_1, using eq. (5.23) for p^s and assuming a fixed SF $S_2^f\left(p^f\right)$ for firm 2. If on the other hand p^s should increase, then the movement along firm 1's SF $\Sigma_1^s\left(p^s;q_1^f,q_2^f\right)$ in the direction of increasing quantity reinforces the rightward shift in this function, resulting unambiguously in increased q_1^s.

exactly the effect of $d\delta_1$ on the conditional expectation of p^s, (2) fixing the parameter vector Θ, and (3) specifying the equilibrium selection rule. It appears unlikely, however, that firm 2's best response to $d\delta_1$ would be to maintain $S_2^f\left(p^f\right)$ fixed as it loses market share in both markets. Rather, as suggested by the forward market equilibria examined in chapter 7,[320] firm 2 will likely want to increase its forward market quantity at most, if not all, prices p^f in response to $d\delta_1$. Accordingly, we may approximate firm 2's optimal response to the increment $d\delta_1$ by a similar positive increment $d\delta_2$ in firm 2's forward market SF $S_2^f\left(p^f\right)$.[321] Such an increment $d\delta_2$ also ultimately shifts firm 2's spot market SF $\Sigma_2^s\left(p^s;q_2^f,q_1^f\right)$ to the right.

At some point, naturally, rightward shifts in both firms' forward market SFs will drive down prices in both markets[322] to a point beyond which further increases in forward market quantity are not profitable for either firm.[323] At this point, firms' forward market

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[320] Namely, even for the markedly asymmetric supplier firms studied in this work, the base case SFs $\overline{S}_i^f\left(p^f\right)$ for each firm tended to approach each other as p^f increased, even for disparate initial quantities. Moreover, in each of the comparative statics test cases, each firm's SF moved in the same direction in response to parameter perturbations at almost all price levels of interest. See Table E.1.

[321] While we assume for simplicity that the approximation $d\delta_2$ is constant for all p^f, the exact optimal response of firm 2 need not, of course, be constant with p^f.

[322] Note that when *both* firms increase their forward market quantities, both spot market SFs—and hence also the aggregate spot market SF—shift outward. As a result, the effect on the spot market-clearing price is then unambiguous: p^s decreases.

[323] From eq. (C.9) in Appendix C, this optimal point (for either firm) is where the derivative of forward market revenue with respect to p^f and marginal expected optimal provisional spot market profits sum to zero.

SFs are the *equilibrium* forward market SFs $\overline{S}_1^f\left(p^f\right)$ and $\overline{S}_2^f\left(p^f\right)$ from the subgame perfect Nash equilibrium derived in chapter 4.

The key results from the above discussion are the inequalities (8.2) and (8.3) indicating the opposing effects on firms' spot market quantities of a rightward shift in $S_1^f\left(p^f\right)$ (and analogously for a rightward shift in $S_2^f\left(p^f\right)$). Consistent with the opposite signs of these effects, we could characterize the strategic interactions discussed above as a "battle for expected market share" in the spot market, waged with forward market SFs. Moreover, firms' market shares in the forward market are obviously also affected by firms' relative aggressiveness in forward market bidding. Similar to the results of Allaz and Vila (1993, 3), therefore, the potential in the multi-settlement SFE model for forward trading by both firms leads to a *prisoners' dilemma* effect: each firm has an incentive to trade in the forward market, but when both firms do so, both end up worse off in that their price-cost margins are smaller. This effect is due only to the potential for forward trading by the duopolists, and is independent of the particular behavioral assumptions in either the forward or spot markets, *provided that* such assumptions do not suppress forward trading itself.[324]

Over the relevant range of prices $p^f \in [0, 2{,}500]$ \$/MWh for the California PX, chapter 7's equilibria showed that strictly increasing forward market SFs yield positive forward market quantities. That is, for the numerical examples of the multi-settlement

[324] As is the case for Green's (1999a, 115) finding that a firm having Cournot conjectures in the forward market and using affine SFs in the spot market will sell no forward contracts.

SFE model examined here, firms' optimal behavior corresponds to *short positions* in the forward market.[325]

It is instructive to note that the output-enhancing property (which translates, generally, to increased aggregate welfare) of forward contracts does not rely on these being *vesting contracts*, that is, contracts whose terms and conditions are subject to regulatory control. Rather, allocating forward contracts via a market-based mechanism—as in the forward market of the multi-settlement SFE model—is sufficient to realize welfare benefits from such contracts. Moreover, in this modeling framework, imposing price controls on forward contracts would lead (if such controls are binding) to smaller forward market positions by suppliers. This outcome, in turn, would result in a lower level of expected aggregate spot market output than in the absence of such price controls. This suggests, further, that such regulation of forward market contracts may lead to lower levels of aggregate welfare.[326] This argument, of course, does not militate against possible distributional rationales for regulatory intervention in the forward market. For example, unregulated forward market prices that are significantly higher than expected spot market prices would create large transfers from consumers to producers, which may be politically undesirable.

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[325] The finding that short positions in the forward market are optimal for suppliers—i.e., $\overline{q}_i^f > 0$ —may be contingent on our choice of base case parameter vector Θ^{base}; section 7.6's comparative statics analysis investigated model solutions within a small neighborhood of this vector. Outside of this region of the parameter space, we may find that we select forward market SFs such that $\overline{q}_i^f < 0$. Given that we restricted our attention to strictly increasing SFs (in the upper partition, such SFs lie in Region I), this result does not appear to be sensitive to the properties of the equilibrium selection procedure for the forward market. As Figure 7.15 suggests, Region I is contained within the positive orthant of the phase space.

[326] A more sophisticated analysis of the welfare effects of forward contracting would require a dynamic analysis in a repeated game setting.

8.1.2 *Effect of a supplier's forward market activity on its rival's profits*

This subsection demonstrates how forward market activity by a supplier *decreases* its rival's profits. We use the same technique as in subsection 8.1.1 of a perturbation (not necessarily in equilibrium) of a firm's forward market SF. Rather than quantity effects, however, the question of interest here is the effect of the SF perturbation on the rival firm's profits. For concreteness, consider a shock $\delta_2 > 0$ to firm 2's forward market SF. Define a base forward market SF $\underline{S}_2^f\left(p^f\right) \equiv S_2^f\left(p^f\right) - \delta_2$ for firm 2, so that

$$S_2^f\left(p^f\right) = \underline{S}_2^f\left(p^f\right) + \delta_2 . \tag{8.4}$$

Given the decomposition of $S_2^f\left(p^f\right)$ in eq. (8.4), a differential shock $d\delta_2$ to firm 2's forward market SF bid translates $S_2^f\left(p^f\right)$ to the right. We examine the effects of this shock to $S_2^f\left(p^f\right)$ on *firm 1's total profits* π_1^{tot*}, whereby this denotes firm 1's *optimal* (but not necessarily *equilibrium*) *profits*. In what follows we assume, naturally, that firm 1 imputes to firm 2 the SF $S_2^f\left(p^f\right)$ in eq. (8.4), and in addition imputes the equilibrium spot market SF $\overline{\Sigma}_2^s\left\{p^s; S_2^f\left(p^f\right), \left[D^f\left(p^f, \varepsilon_0^f\right) - S_2^f\left(p^f\right)\right]\right\}$.

Adapting the forward market optimization problem for the imputation (8.4), we may combine eqs. (4.16)–(4.18) to express π_1^{tot*} as a function of $\underline{S}_2^f\left(p^f\right)$, δ_2, and ε_0^f as follows:

$$
\begin{aligned}
&\pi_1^{tot*}\left\{\underline{S}_2^f(\bullet),\delta_2,\varepsilon_0^f\right\} \\
&\qquad = \max_{p^f}\Big[p^f\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right] \\
&\qquad\qquad + \mathrm{E}\Big(\overline{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right],\underline{S}_2^f\left(p^f\right)+\delta_2,\varepsilon^s\right\}\Big|\,\varepsilon_0^f\Big)\Big],
\end{aligned}
\tag{8.5}
$$

where

$$
\begin{aligned}
&\overline{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right],\underline{S}_2^f\left(p^f\right)+\delta_2,\varepsilon^s\right\} \\
&\qquad = \max_{p^s}\overline{\pi}_1^s\Big\{p^s,\overline{\Sigma}_2^s\left\{p^s;\underline{S}_2^f\left(p^f\right)+\delta_2,\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right]\right\}, \\
&\qquad\qquad \left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right],\varepsilon^s\Big\}
\end{aligned}
\tag{8.6}
$$

and

$$
\begin{aligned}
&\overline{\pi}_1^s\Big\{p^s,\overline{\Sigma}_2^s\left\{p^s;\underline{S}_2^f\left(p^f\right)+\delta_2,\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right]\right\}, \\
&\quad \left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right],\varepsilon^s\Big\} \\
&\qquad = p^s\cdot\Big(D^s\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^s\left\{p^s;\underline{S}_2^f\left(p^f\right)+\delta_2,\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right]\right\}\Big) \\
&\qquad\quad - p^s\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right] \\
&\qquad\quad - C_1\Big(D^s\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^s\left\{p^s;\underline{S}_2^f\left(p^f\right)+\delta_2,\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right]\right\}\Big).
\end{aligned}
\tag{8.7}
$$

Taking the derivative of eqs. (8.5) with respect to δ_2 (using eqs. (8.6) and (8.7) and the envelope theorem), we get

$$
\begin{aligned}
&\frac{\partial\pi_1^{tot*}\left\{\underline{S}_2^f(\bullet),\delta_2,\varepsilon_0^f\right\}}{\partial\delta_2} \\
&\qquad = -p^f+\mathrm{E}\left(-\frac{d\overline{\Sigma}_2^s\{\cdots\}}{d\delta_2}p^s+p^s-C_1'\left(\overline{q}_1^s\right)\left[-\frac{d\overline{\Sigma}_2^s\{\cdots\}}{d\delta_2}\right]\middle|\,\varepsilon_0^f\right),
\end{aligned}
\tag{8.8}
$$

where we have abbreviated the arguments of $\overline{\Sigma}_2^s$ in eq. (8.8) as "$\cdots$."

Under the assumptions of the simplified affine example, we may evaluate the derivative $d\bar{\Sigma}_2^s\{\cdots\}/d\delta_2$ in eq. (8.8) using eq. (5.13) as

$$\frac{d\bar{\Sigma}_2^s\{\cdots\}}{d\delta_2}=\frac{d\bar{\Sigma}_2^s\left\{p^s;\underline{S}_2^f\left(p^f\right)+\delta_2,\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right]\right\}}{d\delta_2}=\phi_2\,. \tag{8.9}$$

Using eq. (8.9) and conditioning instead on a market-clearing price $p^f=p^{f*}\left(\varepsilon_0^f\right)$, we may rearrange eq. (8.8) as

$$\begin{aligned}&\frac{\partial\pi_1^{tot*}\left\{\underline{S}_2^f(\bullet),\delta_2,\varepsilon_0^f\right\}}{\partial\delta_2}\\&\qquad=-\phi_2\left[\mathrm{E}\left(p^s\middle|p^f\right)-\mathrm{E}\left(C_1'\left(\bar{q}_1^s\right)\middle|p^f\right)\right]-p^f+\mathrm{E}\left(p^s\middle|p^f\right).\end{aligned} \tag{8.10}$$

As in the derivation of eq. (5.37) (which relied on the simplified affine example), we may write the expected price-cost margin $\mathrm{E}\left(p^s\middle|p^f\right)-\mathrm{E}\left(C_1'\left(\bar{q}_1^s\right)\middle|p^f\right)$ in eq. (8.10) as

$$\mathrm{E}\left(p^s\middle|p^f\right)-\mathrm{E}\left(C_1'\left(\bar{q}_1^s\right)\middle|p^f\right)=\phi_1\left[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{01}+c_1S_1^f\left(p^f\right)\right)\right]. \tag{8.11}$$

Using eq. (8.11), we may then recast eq. (8.10) as

$$\begin{aligned}&\frac{\partial\pi_1^{tot*}\left\{\underline{S}_2^f(\bullet),\delta_2,\varepsilon_0^f\right\}}{\partial\delta_2}\\&\qquad=-\left\{\phi_1\phi_2\left[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{01}+c_1S_1^f\left(p^f\right)\right)\right]+\left[p^f-\mathrm{E}\left(p^s\middle|p^f\right)\right]\right\}.\end{aligned} \tag{8.12}$$

While the sign of the right-hand side of eq. (8.12) depends on the particular SFs $S_i^f\left(p^f\right)$ selected in the forward market,[327] we are able to determine this sign for cases of interest by the following argument. First, comparing eqs. (7.1) and (7.41) in the previous chapter, we have that

$$\phi_1\phi_2\left[\mathrm{E}\left(p^s\middle|p^f\right)-\left(c_{01}+c_1\overline{S}_1^f\left(p^f\right)\right)\right]+\left[p^f-\mathrm{E}\left(p^s\middle|p^f\right)\right]=\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{21}\overline{S}^{f++}\left(p^f\right). \quad (8.13)$$

Second, in note 267 of that chapter, we argued that everywhere within the phase space's upper partition on which we focus in this work, the quadratic form $\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{21}\overline{S}^{f++}\left(p^f\right)$ has a positive sign,[328] that is,

$$\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{21}\overline{S}^{f++}\left(p^f\right)>0\,. \quad (8.14)$$

Combining the expressions (8.12)–(8.14), we may conclude that

$$\frac{\partial\pi_1^{tot*}\left\{\underline{S}_2^f\left(\bullet\right),\delta_2,\varepsilon_0^f\right\}}{\partial\delta_2}<0\,, \quad (8.15)$$

which says that an increase in forward market activity by firm 2 *decreases* firm 1's total profits π_1^{tot*} at an optimum.

We may interpret inequality (8.15) using Tirole's (1988) terminology from his

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[327] In addition to the explicit appearance of $S_1^f\left(p^f\right)$ in eq. (8.12), recall from eq. (7.9) that the conditional expected spot market price $\mathrm{E}\left(p^s\middle|p^f\right)$ itself depends on both firms' forward market SFs.

[328] This is because the equation $\overline{S}^{f++}\left(p^f\right)^{\top}\mathcal{Q}_{21}\overline{S}^{f++}\left(p^f\right)=0$ characterizes the singular locus, while the upper partition lies entirely on one side (the "positive" side) of the singular locus.

two-period, two-firm model—modified appropriately[329]—analyzing business strategies and strategic interaction.[330] In period 1 of Tirole's modified model, only firm 2—the incumbent—is present in the market.[331] Firm 2 chooses a variable, which Tirole calls an "investment," denoted as K_2 (K_2 could be productive capacity, for example, although in general K_2 might be any variable affecting period 2 competition). In period 2, firm 1 observes K_2 and decides to either enter, or not to enter, the market. Tirole (1988, 325) classifies competitive scenarios in terms of the effect of firm 2's investment K_2 on firm 1's profits (in the "entry-deterrence"[332] case). Denoting firm 1's total profits as Π^1, Tirole associates the condition

$$\frac{d\Pi^1}{dK_2} < 0 \tag{8.16}$$

---

[329] In the following account of Tirole's model, we exchange the (arbitrary) subscripts 1 and 2 labeling Tirole's firms—so that *firm 2* is the incumbent—for consistency with the foregoing analysis of the multi-settlement SFE model. The objective here is to show that inequality (8.15) above is consonant with Tirole's analysis in terms of the effect of one firm's first-period (e.g., forward market) action on its rivals' profits.

[330] Tirole's analysis expands on that of Fudenberg and Tirole (1984) and Bulow, Geanakoplos, and Klemperer (1985). The classic example of such a model is a two-period entry deterrence/accommodation game between an incumbent and a potential entrant, but the problem's basic structure applies to a considerable range of interesting economic problems; in addition to Tirole (1988), see Bulow, Geanakoplos, and Klemperer (1985) for many other examples.

[331] The presence of only a *single* incumbent firm in period 1 is a critical distinction between Tirole's model and the multi-settlement SFE model of this thesis, in which (as thesis chapter 7's various numerical examples show) each firm is active in *both* the forward and spot markets. In our setting, both firms can and do make strategic choices in period 1, and hence the incumbent-entrant distinction is not relevant in the multi-settlement SFE model. See also note 329 regarding the labeling of the two firms in this discussion of Tirole's model.

[332] Denote firm 1's total profits as Π^1. For the incumbent firm 2 to (just) deter firm 1's entry, firm 2 chooses K_2 so that $\Pi^1 = 0$. Hence, in the case of entry deterrence in Tirole's model, it is the effect of the incumbent's period 1 action on the potential entrant's total profits that determines the entry decision.

with the case in which investment (i.e., increasing K_2) makes firm 2 "tough"[333] in Tirole's terminology. If, on the other hand, we have that

$$\frac{d\Pi^1}{dK_2} > 0, \tag{8.17}$$

Tirole characterizes this situation as the case of investment making firm 2 "soft."[334]

Making the analogy between Tirole's framework and the present multi-settlement SFE model (and arbitrarily taking firm 2 to be the "incumbent" in the latter model—see note 331), it is natural to view firm 2's forward market activity $\underline{S}_2^f\left(p^f\right)+\delta_2$ as analogous to an investment for that firm, to use Tirole's terminology. Appealing to this analogy, we see from inequalities (8.15) and (8.16) above that the profit derivatives $\partial\pi_1^{tot*}\left\{\underline{S}_2^f\left(\bullet\right),\delta_2,\varepsilon_0^f\right\}\big/\partial\delta_2$ and $d\Pi^1\big/dK_2$ correspond to each other. In particular, note that the *sign* of both of these derivatives is negative. These observations suggest that we may also apply Tirole's terminology to the multi-settlement SFE model. Namely, we could interpret the negative effect on firm 1's profits (i.e., inequality (8.15)) as *the forward market action* $\underline{S}_2^f\left(p^f\right)+\delta_2$ *making firm 2 tough* (or, recalling note 333, "*disadvantaging firm 1*"). This is consistent with the intuition from previous chapters that increasing δ_2 (ceteris paribus) shifts firm 2's spot market SF $\overline{\Sigma}_2^s\left\{p^s;\underline{S}_2^f\left(p^f\right)+\delta_2,\left[D^f\left(p^f,\varepsilon_0^f\right)-\underline{S}_2^f\left(p^f\right)-\delta_2\right]\right\}$ to the right, thereby making that firm

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[333] Perhaps more evocatively, we might instead characterize $d\Pi^1\big/dK_2 < 0$ (inequality (8.16)) as a situation of the investment *K* "disadvantaging one's competitor."

[334] Conversely to note 333, we could instead say that $d\Pi^1\big/dK_2 > 0$ (inequality (8.17)) exemplifies the case of the investment *K* "favoring one's competitor."

more aggressive in the spot market in that it bids a larger quantity at each price. Naturally, increasing δ_2 makes firm 2 a more aggressive competitor in the *forward market* in the same sense, as well.

Competition in supply functions is distinct, naturally, from competition in quantities à la Cournot. Nevertheless, the multi-settlement SFE model also reflects Tirole's (1988, 336) generalization that two-period *quantity* games are often more competitive than their static (one-period) counterparts. We see evidence of this more competitive property in the tendency of forward market activity to increase one's own spot market quantity (as in inequality (8.3)), as well as in the analysis of expected welfare of section 7.7. In these welfare computations, we found that expected aggregate welfare of the multi-settlement SFE model exceeded that for the single-settlement SFE model, due, in part, to the larger expected spot market quantities in the multi-settlement model.

8.1.3 Decomposition of suppliers' incentives for forward market activity

In his model, Tirole (1988) emphasizes the role of investments as *commitments*, in particular, "commitments that matter because of their influence on the rivals' actions" (p. 323). Likewise, we may usefully view forward market positions in the multi-settlement SFE model as *strategic commitments*; these similarly influence rivals' actions as we explain below.

In this subsection, we sharpen the focus on strategic considerations and examine in more detail how strategic motives affect firms' forward market decisions. The analysis here is not fundamentally new; rather we simply parse firm 1's forward market equilibrium optimality condition (5.37) in a new way. Namely, we decompose a version of firm 1's forward market equilibrium optimality condition so as to emphasize firm 2's impact, via each market, on firm 1's forward market action.

We begin by rewriting firm 1's forward market equilibrium optimality condition (5.37) as eq. (8.18) below:

$$\begin{aligned}\Big\{\phi_1\phi_2\Big[\mathrm{E}\big(p^s\big|p^f\big)-\big(c_{01}+c_1\bar{S}_1^f\big(p^f\big)\big)\Big]-\Big[\mathrm{E}\big(p^s\big|p^f\big)-p^f\Big]\Big\}\bar{S}_2^{f\prime}\big(p^f\big)\\ =\bar{S}_1^f\big(p^f\big)-D_0^{f\prime}\big(p^f\big)\Big[\mathrm{E}\big(p^s\big|p^f\big)-p^f\Big].\end{aligned} \tag{8.18}$$

Equation (8.18) is a version of firm 1's first-order necessary condition for its forward market optimization problem (eqs. (4.16)–(4.18)) under the assumptions of the simplified affine example.[335] We may re-introduce the derivative $d\tilde{\pi}_1^{tot}\big(p^f, \tilde{S}_2^f\big(p^f\big)\big)\big/dp^f$, rearrange the terms in eq. (8.18), and decompose the new expression into terms that we label the *direct effect*, the *settlement effect*, and the *strategic effect*, as follows:[336]

---

[335] Equation (8.18) reflects (1) the substitution of $\bar{S}_1^f\big(p^f\big)$ for $D^f\big(p^f, \varepsilon_0^f\big)-\bar{S}_2^f\big(p^f\big)$ from the market-clearing condition for the forward market, as well as (2) the substitution of $D_0^{f\prime}\big(p^f\big)$ for $D^{f\prime}\big(p^f, \varepsilon_0^f\big)$ from eq. (3.13). Recall also that in the simplified affine example, firms' marginal cost functions and spot market SFs as well as spot market demand functions all possess affine functional forms.

[336] The reader is warned that the direct effect and the strategic effect defined in eq. (8.19) for the multi-settlement SFE model are in the same spirit as—but *distinct from*—the "direct effect" and "strategic effect" identified in Fudenberg and Tirole (1984, 363) and later, Tirole (1988, sec. 8.3). In the present work, we define these effects via differentiation with respect to a price p^f in eq. (8.19), consistent with the derivation of the SFs. The other authors cited motivate the definition of these effects by differentiating with respect to a (firm-specific) quantity.

$$\frac{d\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right)\right)}{dp^f} = \underbrace{\bar{S}_1^f\left(p^f\right) + p^f\left[D_0^{f\prime}\left(p^f\right) - \bar{S}_2^{f\prime}\left(p^f\right)\right]}_{\text{Direct effect}}$$
$$\underbrace{-\mathrm{E}\left(p^s \middle| p^f\right)\left[D_0^{f\prime}\left(p^f\right) - \bar{S}_2^{f\prime}\left(p^f\right)\right]}_{\text{Settlement effect}} \tag{8.19}$$
$$\underbrace{-\phi_1\phi_2\bar{S}_2^{f\prime}\left(p^f\right)\left[\mathrm{E}\left(p^s \middle| p^f\right) - \left(c_{01} + c_1\bar{S}_1^f\left(p^f\right)\right)\right]}_{\text{Strategic effect}}$$
$$= 0.$$

Below, we discuss how each of the three constituent effects in eq. (8.19) shaping firm 1's forward market behavior arises.

We call the term

$$\bar{S}_1^f\left(p^f\right) + p^f\left[D_0^{f\prime}\left(p^f\right) - \bar{S}_2^{f\prime}\left(p^f\right)\right] \tag{8.20}$$

on the right-hand side of eq. (8.19) the *direct effect* since it represents firm 1's response to its forward market residual demand function, $RD_1^f\left(p^f, \varepsilon_0^f\right) \equiv D^f\left(p^f, \varepsilon_0^f\right) - \bar{S}_2^f\left(p^f\right)$, considering the forward market in isolation. Given that firm 1 faces this residual demand function, the expression (8.20) is the derivative of firm 1's forward market revenue, $p^f\left[D^f\left(p^f, \varepsilon_0^f\right) - \bar{S}_2^f\left(p^f\right)\right]$, with respect to p^f (using again the substitutions of note 335).[337]

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[337] Before imposing Nash equilibrium, firm 1 solves its forward market problem, given ε_0^f (as detailed in chapter 4), to yield a—firm-specific—optimal price $p_1^{f*}\left(\varepsilon_0^f\right)$. In Nash equilibrium, naturally (see note 141), firms construct their forward market SFs such that $p_1^{f*}\left(\varepsilon_0^f\right) = p_2^{f*}\left(\varepsilon_0^f\right) \equiv p^{f*}\left(\varepsilon_0^f\right)$, which we denote in equilibrium as simply the forward market price p^f. Consistent with previous chapters' conventions, we interpret eq. (8.19) and the associated analysis as applying to such an *equilibrium* outcome, though we could just as well recast the above discussion in terms of firm 1's optimal—though not necessarily equilibrium—price $p_1^{f*}\left(\varepsilon_0^f\right)$.

The *settlement effect*

$$-\mathrm{E}\left(p^s \middle| p^f\right)\left[D_0^{f\prime}\left(p^f\right)-\bar{S}_2^{f\prime}\left(p^f\right)\right] \tag{8.21}$$

on the right-hand side of eq. (8.19) is the expected change in firm 1's settlement payment $p^s q_1^f$ made in the spot market for a marginal change in p^f, again given its forward market residual demand function $RD_1^f\left(p^f,\varepsilon_0^f\right)$. The settlement effect depends on the expected (optimal) spot market price $\mathrm{E}\left(p^s \middle| p^f\right)$, conditional on p^f.

Finally, the *strategic effect*

$$-\phi_1\phi_2\bar{S}_2^{f\prime}\left(p^f\right)\left[\mathrm{E}\left(p^s \middle| p^f\right)-\left(c_{01}+c_1\bar{S}_1^f\left(p^f\right)\right)\right] \tag{8.22}$$

on the right-hand side of eq. (8.19) arises due to the conjectured *spot market* response of firm 2 to firm 1's choice of (optimal) price p^f $\left(=p_1^{f*}\left(\varepsilon_0^f\right)\right)$. Looking back at chapter 4's analysis, we may show that the strategic effect of eq. (8.22) is simply the expression $\psi_i\left(p^f\right)$ in eq. (4.42) under the assumptions of the simplified affine example. As observed in Appendix C, we may interpret $\psi_i\left(p^f\right)$, in turn, as the expected change in the difference between firm 1's equilibrium spot market revenue and production cost (evaluated at its equilibrium contract quantity $\bar{S}_1^f\left(p^f\right)$) for a marginal change in p^f.

To understand how the strategic effect arises, begin by rewriting eq. (5.13) for firm 2 as eq. (8.23) below:

$$\frac{\partial\bar{\Sigma}_2^s\left(p^s;\bar{q}_2^f,\bar{q}_1^f\right)}{\partial\bar{q}_2^f}=\phi_2\,. \tag{8.23}$$

Equation (8.23) expresses the marginal effect of changes in firm 2's equilibrium forward market quantity $\overline{q}_2^f$ (ceteris paribus) on its spot market quantity for an arbitrary price p^s.[338] Firm 2's spot market SF $\overline{\Sigma}_2^s\left(p^s;\overline{q}_2^f,\overline{q}_1^f\right)$ appears in firm 1's spot market residual demand function

$$q_1^s = D^s\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^s\left\{p^s;\overline{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\overline{S}_2^f\left(p^f\right)\right]\right\} \tag{8.24}$$

in problem (4.18) (taking $\tilde{S}_2^f\left(p^f\right)=\overline{S}_2^f\left(p^f\right)$, in equilibrium). Firm 1 conjectures that, in equilibrium, firm 2 responds to marginal changes in p^f according to the function $\overline{S}_2^f\left(p^f\right)$. Firm 1 induces firm 2 in this way to change $\overline{q}_2^f$ (and hence $\overline{q}_2^s$), as already described in subsection 8.1.1 above. This is the heart of the strategic effect. With these considerations, eqs. (8.23) and (8.24) imply that the change in q_1^s for a marginal change in p^f given the functions $\overline{S}_2^f\left(p^f\right)$ and $\overline{\Sigma}_2^s\left\{p^s;\overline{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\overline{S}_2^f\left(p^f\right)\right]\right\}$ and holding p^s fixed is

$$\left.\frac{dq_1^s}{dp^f}\right|_{\overline{S}_2^f,\,\overline{\Sigma}_2^s,\,p^s}=\left(\left.\frac{\partial q_1^s}{\partial\overline{q}_2^s}\right|_{p^s}\right)\left(\left.\frac{\partial\overline{\Sigma}_2^s}{\partial\overline{q}_2^f}\right|_{\overline{\Sigma}_2^s,\,p^s}\right)\left(\left.\frac{d\overline{q}_2^f}{dp^f}\right|_{\overline{S}_2^f}\right)=-\phi_2\overline{S}_2'^f\left(p^f\right), \tag{8.25}$$

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[338] Recall that the simplicity of the (constant) expression for the derivative in eq. (8.23) depended critically on the assumption of $\overline{\Sigma}_2^s\left(p^s;\overline{q}_2^f,\overline{q}_1^f\right)$ being affine in p^s. In the affine case, $\overline{\Sigma}_2^s\left(p^s;\overline{q}_2^f,\overline{q}_1^f\right)$ is, in fact, independent of $\overline{q}_1^f$. This suggests that if we extended the investigation to include *non-affine* SFs $\overline{\Sigma}_j^s\left(p^s;\overline{q}_j^f,\overline{q}_i^f\right)$, we would observe an additional term in the strategic effect for firm *i*. This term would correspond to a shift in $\overline{\Sigma}_j^s\left(p^s;\overline{q}_j^f,\overline{q}_i^f\right)$ due to $\overline{q}_i^f=D^f\left(p^f,\varepsilon_0^f\right)-\overline{S}_j^f\left(p^f\right)$ also changing with p^f.

as reflected in the expression (8.22) for the strategic effect. Finally, we note that the strategic effect is proportional[339] to firm 1's (conditional) expected price-cost margin

$$\mathrm{E}\left(p^s \middle| p^f\right) - \left(c_{01} + c_1 \overline{S}_1^f\left(p^f\right)\right)$$

in the spot market, evaluated at its equilibrium forward market quantity $\overline{q}_1^s = \overline{q}_1^f = \overline{S}_1^f\left(p^f\right)$.

If we rewrite firm 1's FOC, eq. (8.19), setting the strategic effect to zero, the revised FOC assuming "zero strategic effects"[340] is as follows:

$$\begin{aligned} \left.\frac{d\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right)\right)}{dp^f}\right|_{\substack{\text{Zero strategic} \\ \text{effects}}} &= \underbrace{\overline{S}_1^f\left(p^f\right) + p^f\left[D_0^{f\prime}\left(p^f\right) - \overline{S}_2^{f\prime}\left(p^f\right)\right]}_{\text{Direct effect}} \\ &\quad \underbrace{-\mathrm{E}\left(p^s \middle| p^f\right)\left[D_0^{f\prime}\left(p^f\right) - \overline{S}_2^{f\prime}\left(p^f\right)\right]}_{\text{Settlement effect}} \qquad (8.26) \\ &= 0. \end{aligned}$$

A full accounting of the influence of the strategic effect on the forward market equilibrium would require computing new forward market SFs from the FOC (8.26) (and the symmetric condition for firm 2). In particular, the functions $\mathrm{E}\left(p^s \middle| p^f\right)$ and $D^f\left(p^f, \varepsilon_0^f\right)$, derived analytically and computed numerically in chapters 6 and 7, both depend endogenously on the functions $\overline{S}_i^f\left(p^f\right)$.

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339 With constant of proportionality $-\phi_1\phi_2\overline{S}_2^{f\prime}\left(p^f\right)$ (given p^f), from the expression (8.22).

340 This would be the case if the derivative $\partial\overline{\Sigma}_2^s\left(p^s; \overline{q}_2^f, \overline{q}_1^f\right) \big/ \partial\overline{q}_2^f$ from eq. (8.23) and its analog for firm 1 were equal to zero, implying that $\phi_1 = \phi_2 = 0$.

8.1.4 *Motives for forward market activity by consumers*

To conclude this section, we turn briefly to the demand side of the market to consider *consumers'* motives for forward market participation in the multi-settlement SFE model. In contrast to the treatment of suppliers, we take consumers to be risk averse in this model. Accordingly, consumers pay a risk (or hedging) premium to suppliers in purchasing forward contracts at a price p^f typically in excess of $\mathrm{E}\left(p^s \middle| p^f\right)$. Hedging is thus a motive for consumers' participation in the forward market, as chapter 6 analyzed in detail. Speculative motives also exist for consumers in the forward market. Assuming—for a representative consumer *R*—a forward contract quantity q_R^f, the conditional expectation of the term $\left(p^s - p^f\right)q_R^f$ enters the objective function of her nested maximization problem (6.30) (letting $j = R$) for the multi-settlement SFE model. The presence of this term indicates that the representative consumer *R* speculates on the (conditional) expected price difference $p^s - p^f$. Finally, because consumers take prices as parametric in problem (6.30), they do not behave strategically in the present model. Therefore, consumers face no strategic motives for forward market participation.

8.2 Further research: Relaxing restrictions imposed in the model

8.2.1 *Number of competitors* n

Throughout this investigation, we have assumed that we have a duopoly on the supply side of the market—that is, $n = 2$. In this section, we consider what increasing *n* would entail, and how the results of the multi-settlement SFE model might be affected.

The equilibrium optimality conditions for the case of larger *n* are easy to express. The major structural change is to replace (in firm *i*'s optimization problem) the SFs

$\overline{S}_j^f\left(p^f\right)$ and $\overline{\Sigma}_j^s\left(p^s;\overline{q}_j^f,\overline{q}_i^f\right)$ with the sums $\sum_{j\neq i}\overline{S}_j^f\left(p^f\right)$ and $\sum_{j\neq i}\overline{\Sigma}_j^s\left(p^s;\overline{q}_j^f,\overline{q}_i^f\right)$, respectively, reflecting the actions of *all* of firm *i*'s rivals in *i*'s residual demand functions for each market. The difficulties that are likely to arise for larger *n* appear to be largely computational. Preliminary investigations show that our chosen numerical analysis package, MATLAB (and the MAPLE symbolic algebra kernel),[341] has difficulty solving the *spot market* problem[342] symbolically in the affine case for $n>6$. This may be because a solution simply does not exist, because the problem is ill-posed given the solver's algorithm, or that the solver—as currently configured—is unable to solve it. Numerical solutions, in contrast, may of course be easier to find.

We have not yet attempted to find solutions of the forward market problem for $n>2$ firms. While visualization of such trajectories becomes more difficult for such larger *n*, we anticipate no fundamental obstacle to applying the MATLAB- or Excel-based models to cases of larger *n*. Obtaining a solution for a larger value of *n* would offer a considerable improvement in verisimilitude over the current duopoly case in view of the structure of actual electricity markets. Moreover, a larger *n* would permit modeling of various scenarios encompassing firm entry, exit, generation plant divestiture, and mergers and acquisitions. Finally, if we are able to model cases in which *n* gets large, it would be interesting to see whether spot and forward market SFs approach a competitive limit. We might obtain some insight into the nature of such a competitive limit from the symmetric—and computationally far simpler—case with *n* identical firms.

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[341] The MathWorks (2001) and Maplesoft (2002).

[342] In the affine case, solving this problem entails solving a nonlinear system of algebraic—not differential—equations.

8.2.2 Affine functional form restrictions

Beginning with the simplified affine example of chapter 5, much of the analysis is restricted to the case in which the spot market demand function, both firms' marginal cost functions, and spot market SFs have an affine functional form. Suppose, in contrast, that we broaden the focus from affine spot market SFs to consider also non-affine SFs arising from the spot market equilibrium optimality conditions, eqs. (4.13) and (4.14). In this case, however, these conditions would no longer yield a system of simultaneous algebraic equations for spot market SF slopes (eqs. (5.6) and (5.11)); instead, a differential equation system[343] will characterize the spot market SFs. Because in this case the spot market SFs will no longer be unique, issues of equilibrium selection and coordination between firms would arise in the spot market as well, leading to the compound problem that Newbery (1998, 733) has characterized as a "double infinity of solutions."[344] Failure of the firms to coordinate successfully on an equilibrium would not necessarily lead to market instability, but it would imply that firms are almost certainly not supplying ex post optimal quantities, given their rival's actions and realizations of stochastic parameters. Provided that the firms realize that their behavior is suboptimal, we could surmise that they might engage in a heuristic search process in their strategy spaces in an attempt to improve their profits.

We could go further in generalizing the affine case, and assume only strictly increasing marginal cost and downward-sloping spot market demand. Under these

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[343] Similar to the forward market problem in the present version of the multi-settlement SFE model.

[344] Other criteria for equilibrium selection (e.g., Pareto optimality, rationalizability—recall n. 123—might be invoked, but the theory here is generally inconclusive and somewhat controversial (see Fudenberg and Tirole 1991, 48–53).

assumptions, not only would we again have a continuum of nonlinear equilibrium SFs in the spot market, but the existence of an affine equilibrium in spot market SFs would not be assured. Nonetheless, given any differentiable marginal cost and demand functions, it is straightforward as an analytical matter to generalize the (necessary) equilibrium optimality conditions for both markets. Then, assuming some procedure for equilibrium selection in each market, it should be possible to solve the resulting systems of ODEs numerically using the methods of chapter 7. Extreme functional forms, however—for example, demand that is too convex, or marginal costs that are too steep or nonconvex—may cause a multi-settlement market equilibrium with strictly increasing forward market SFs not to exist, or to exist only on a sharply restricted price domain or region of the parameter space.

8.2.3 Role of perfect observability of forward market actions

Beginning with the specification of the multi-settlement SFE model's information structure in subsection 3.1.1, we have assumed throughout this thesis that firms' equilibrium forward market actions—that is, the SFs $\overline{S}_i^f\left(p^f\right)$—are *perfectly observable* as they formulate their spot market SFs. In section 3.3, however, we showed that firm *i*'s spot market SF is a function of the spot market price, p^s, and also (in general) each firm's forward market quantity, q_i^f ($i = 1, 2$). Hence, we wrote firm *i*'s (equilibrium) spot market SF as a function $\overline{\Sigma}_i^s\left(p^s; \overline{q}_i^f, \overline{q}_j^f\right)$ ($i, j = 1, 2;\ i \neq j$).[345] From this specification, it is clear that we may weaken the observability assumptions from

[345] Where firms' respective equilibrium forward market quantities coincide with the optimal as well as the imputed quantities, so that we may write $\overline{q}_i^f = q_i^f = \tilde{q}_i^f$.

observing forward market *actions* (the SFs $\overline{S}_i^f\left(p^f\right)$) to simply observing forward market equilibrium *quantities* $\overline{q}_i^f$. Observability of the $\overline{q}_i^f$ is crucial to the model, however. Weakening the model's assumptions further in this respect by permitting less-than-perfect observability of $\overline{q}_i^f$ would likely have a critical effect on the model's results, particularly on the strategic incentives that arise between markets as discussed in subsection 8.1.3 above. In this subsection, we examine some related literature that suggests how introducing imperfect observability of forward market outcomes might affect solutions of the multi-settlement SFE model.

Hughes and Kao (1997) study a two-stage Cournot duopoly game with forward contracting in the first stage and production in the second stage. The authors examine how observability of contract positions affects strategic and hedging motives for forward contracting; they consider cases in which the competitors are risk neutral, and in turn, risk averse. Table 8.1 below summarizes Hughes and Kao's main results.

TABLE 8.1 MOTIVES FOR FORWARD MARKET PARTICIPATION AS A FUNCTION OF OBSERVABILITY OF CONTRACT POSITIONS AND RISK PREFERENCES (Hughes and Kao 1997)

Risk preferences / *Observability of contract positions*	*Risk neutral*	*Risk averse*
Perfectly observable	Strategic motive only	Hedging motive Strategic motive
Not observable	No forward contracting[a]	Hedging motive Strategic motive[b]

Notes:

[a] Hughes and Kao require conjectures to be consistent with firms' actions; accordingly, the only consistent conjecture for the forward market in this case is $\tilde{q}_1^f = \tilde{q}_2^f = 0$.

[b] Here, the strategic motive is weaker than that in the cell above.

The intuition for the absence of forward contracting in the (Not observable, Risk neutral) cell of Table 8.1 is as follows, letting i and j index the two firms ($i, j = 1, 2;\ i \neq j$) (Hughes and Kao 1997, 125): If firm j conjectures that firm i does not take a forward position, then firm i has no incentive to deviate from this conjecture. In essence, absent observability, there is no means for firm i to alter firm j's beliefs. Finally, the presence of a strategic motive for forward market activity in the (Not observable, Risk averse) cell of Table 8.1 bears some explanation. To see that a strategic motive is present in this case despite unobservable contract positions, note that the risk-averse firm i is aware that firm j expects it to *hedge* price uncertainty via forward contracts. As a consequence, firm j's residual demand function shifts to the left, causing firm j to concede market share in the second stage. In this way, hedging behavior can have strategic consequences.[346]

Consider now the implications of the above findings for the multi-settlement SFE model using supply functions. The behavioral assumption of supply functions invoked in the present work is significantly more flexible than the Cournot conjectures used by Hughes and Kao. The slope of firm j's imputed forward market SF at an arbitrary price p^f, $\tilde{S}_j^{f\prime}\left(p^f\right)$, is firm i's conjecture regarding how firm j would respond (locally, near p^f) to a change in forward market price. Under our assumed market rules, this slope may lie anywhere on the positive real line (including zero). This significantly greater degree of flexibility may permit consistent conjectures in the SFE case where this was not

[346] Allaz (1987, 42, n. 43) also alluded to this phenomenon when he observed that strategic and hedging motives "partly overlap."

possible under Cournot.[347] Without more careful investigation we cannot be sure, but there is sufficient reason to be skeptical that the "No forward contracting" result in the (Not observable, Risk neutral) cell of Table 8.1 will also obtain under the SFE assumption used in the multi-settlement SFE model.

Rather than positing uncertainty in demand as we do in the present work, Hughes and Kao suppose (in one part of their analysis) that a pair of Cournot duopolists face uncertainty in their costs that is resolved after the forward market clears, but before firms act in the spot market. In this setting, the authors consider in turn the cases of perfectly observable forward contract positions, and unobservable contract positions. For the case of *perfectly observable* contract positions, the authors make a further distinction with regard to risk preferences. For firms that are risk neutral to moderately risk averse, selling forward contracts is optimal, whereas if firms' risk aversion is sufficiently great, firms buy forward contracts.[348] For *unobservable* forward contract positions, on the other

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[347] As is well-known, the Cournot model has inconsistent conjectures. In contrast, given constant marginal cost, the Bertrand model has consistent conjectures. See Bresnahan (1981) for details.

[348] Here, the intuition is that the strategic and hedging effects facing each firm go in *opposite directions*. The firm's risky costs create variance in its profits. This motivates the firm to decrease production, which it can do, effectively, by *buying* forward contracts. On the other hand, the firm's quantity decisions are made, naturally, based on this uncertain cost; these quantity decisions, in turn, affect price, making price risky as well. This effect on price creates an incentive for the risk-averse firm to *sell* forward contracts to lock in sales at a certain price. Unlike the aforementioned effect on cost, this effect on price also naturally affects the profits of the rival, so that the price risk entails a *strategic* as well as a hedging component. When the firm's risk aversion is sufficiently low, the strategic effect dominates, and the firm sells forward contracts. When on the other hand the firm's risk aversion is sufficiently high, hedging is the dominant effect, and the firm buys forward contracts.

With respect to hedging, this behavior is borne out on the demand side of the multi-settlement SFE model. Specifically, subsection 7.6.3 discusses the effect of increasing the representative consumer's risk aversion coefficient λ_R on the forward market equilibrium. There, we noted that increased λ_R causes $D^f\left(p^f, \varepsilon_0^f\right)$ to shift to the right, that is, increased consumer risk aversion increases the demand for forward contracts.

hand, Hughes and Kao find that the strategic effects—while present—are sufficiently attenuated[349] so that, provided only that the firm is risk averse, it buys forward contracts.

Beyond the effects of contracting, there is a wider literature on the critical role of *observability* on strategic incentives in dynamic games that is relevant to our model. Interestingly, some authors have generalized the simple dichotomy between *perfectly observable* and *unobservable* actions by introducing *noisy observations* of first-period actions, made operational via a random deviation between an observed and an actual parameter value representing agents' first-period actions. Bagwell (1995), for instance, has analyzed two-period Stackelberg games of quantity choice in which the follower's observation of the leader's chosen quantity is noisy. He shows that the pure strategy equilibrium set of this game coincides with that for the corresponding simultaneous-move game (i.e., the Cournot equilibrium), even for a very small degree of noise. The implication of this finding is that imperfect observability can negate the commitment inherent in the leader's action for the second period's stage game. In a related article, Maggi (1999) demonstrates that permitting the leader in the aforementioned game to observe private information (e.g., its own cost) generally restores the Stackelberg outcome.[350] Maggi (1999, 556) provides the intuition for this result. Suppose that the leader's private information concerns its type, for example, low-cost or high-cost. The follower will then use the (noisy) observation of the leader's quantity to attempt to infer the leader's type, whereby both pooling and separating equilibria are, in general, possible. Given that the follower behaves in this way, the leader then has an incentive to

[349] Recall the discussion of Table 8.1 above.

[350] See Maggi (1999) for some technical qualifications to this result.

manipulate the follower's perception of the leader's type through its quantity choice. Specifically, the leader's incentive is to produce more than the Cournot output, which restores the Stackelberg first-mover advantage.

Returning to the model in the present work, the multi-settlement market studied here is distinct from the Stackelberg sequential-move setting in that the former model comprises two simultaneous-move stage games. While sequential moves within each stage game would not be a realistic scenario in electricity markets, the sequence of stage games—the forward and spot markets—is a dynamic setting in which the insights of the Stackelberg game apply. Moreover, *imperfect observability* of forward market positions would add realism to our framework. In addition, if we were to revisit the model's information structure, a more plausible assumption would be to have firms' costs be private information. With these changes, the information structure in the multi-settlement SFE model would parallel that in Maggi (1999) discussed in the foregoing paragraph. In this new setting we conjecture that, using an insight similar to that of Maggi, private cost information would offset imperfect observability so that strategic incentives are not impaired by imperfect observability of forward market contract positions.

The above discussion of the observability of forward market contract positions has important policy implications regarding regulatory rules for information disclosure in electricity markets. Given the result of section 7.7's welfare analysis that forward markets are welfare-enhancing,[351] the question arises of how disclosure policies for

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[351] While this result does not hold in all cases, it is generally supported by the previous research discussed in section 1.5.2. For a counter-example, see Ferreira (2003).

forward market positions affect firms' participation in—and hence the welfare effects of—forward markets. In Table 8.1's summary of Hughes and Kao's (1997) model, eliminating information disclosure effectively halts forward trading in the risk-neutral case and attenuates the strategic incentive for forward trading in the risk-averse case. This suggests (but does not prove, as Hughes and Kao point out (p. 130)) that disclosure of forward contract positions will be welfare-enhancing.[352] As noted in the discussion of Table 8.1 above, it appears unlikely that making forward market positions unobservable in the SFE model will completely eliminate forward contracting in the risk-neutral case, although it may still weaken the incentive to contract. If so, then disclosure may not be as critical under the SFE assumption as in the Cournot case.

8.3 Further research: Market power

The present work is only suggestive of the complexities that market power analysis in real-world electricity markets must confront, as Quan and Michaels (2001, 106) attest (with reference to California's market):

> Over the course of a day, a generator must make (by our rough count) at least 480 price bid decisions at various hours. Choosing not to participate in certain markets may require as much thought, and be fraught with as much risk, as choosing to bid in others. Since generators actually bid hourly supply schedules with up to 15 segments in many markets, the potential price decisions over a day run into the thousands. We also count 146 capacity commitment decisions over the day . . . .
>
> The analysis of market power by sellers in a system like this is so complex an endeavor that for all practical purposes it is impossible to perform.

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[352] In more recent work, Hughes, Kao, and Williams (2002) examine the disclosure decision from the firm's perspective in an asymmetric duopoly model in which only one of the two duopolists trades in the forward market. They analyze the tradeoff that disclosure presents to a firm between exploiting its informational advantage in the forward market, on the one hand, and influencing the later production decisions of rivals, on the other. They find, unsurprisingly, that informed firms prefer non-disclosure of forward market positions. In contrast, uninformed market participants (specifically, brokers who are constrained to break even in their trading) prefer disclosure of forward contracts. The authors consider only in passing the symmetric case of competing duopolists in the forward market. They conjecture that symmetry would only strengthen incentives for non-disclosure on the part of informed market participants.

Quan and Michaels' rather pessimistic assessment notwithstanding, the present work has broken new ground in understanding strategic interaction in a multi-settlement market setting. The decomposition and analysis of the incentives for forward market participation is a necessary and important first step toward market power analysis in this novel institutional environment. The next question that we confront in this regard is how the present model might be used or augmented to establish a forward market *perfectly competitive behavioral benchmark* (PCBB) for market power assessment in the presence of risk-averse consumers. The only alternative to forward market participation that we represent in the multi-settlement SFE model is, of course, the opportunity to participate in the spot market. In this setting, it is natural to seek a PCBB in the form of a *marginal opportunity cost* of forward market participation involving expected spot market returns foregone through forward market activity. While we reserve for future research the development of an explicit expression for marginal opportunity cost, we may gain some insight into the nature of such an opportunity cost by revisiting the relevant discussion from chapter 1.

We raised the question in chapter 1 of how to evaluate the competitiveness of the forward market. In particular, we asked whether assessing market power in multi-settlement markets required the *joint* evaluation of behavior in both the forward and spot markets, or whether we could analyze the forward market in isolation. We now revisit this question. In eq. (4.41), we found that the expected spot market price $\mathrm{E}\left(p^s \middle| p^f\right)$ plays the role of marginal production cost $C'(S)$ in eq. (4.15). This structural similarity between the optimality conditions for the forward and spot markets suggests a natural interpretation of the expected spot market price $\mathrm{E}\left(p^s \middle| p^f\right)$ as one contributing factor to the marginal opportunity cost of a supplier's forward market activity. In particular, the settlement effect defined in eq. (8.19) is proportional to the conditional expectation $\mathrm{E}\left(p^s \middle| p^f\right)$. To evaluate $\mathrm{E}\left(p^s \middle| p^f\right)$, naturally, we need to know the conditional distribution of $p^s \middle| p^f$, which in turn requires chapter 5's analysis of how the forward and spot markets are coupled.[353] This is not the entire story concerning marginal opportunity cost, however. From expression (8.19) above, the strategic effect also affects a firm's forward market SF bid through anticipated spot market outcomes.[354] Like the settlement effect, the strategic effect depends on $\mathrm{E}\left(p^s \middle| p^f\right)$ and, in addition, depends on firm 1's expected marginal cost given p^f. Thus, to compute the strategic effect, we again require information from both the forward and spot markets. We may conclude from the definitions of both the settlement and strategic effects in expression (8.19) that evaluating the competitiveness of a firm's forward market behavior requires considering behavior and market outcomes in *both* the forward and spot markets. Further research should define more precisely the marginal opportunity cost of forward market activity in this model, which may then serve as an appropriate PCBB in this two-market setting.

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[353] For these purposes, we developed an expression for $\mathrm{E}\left(p^s \middle| p^f\right)$ in eq. (5.33), later simplified to eq. (7.9).

[354] Expression (8.19) also includes the *direct effect* $\overline{S}_1^f\left(p^f\right) + p^f\left[D_0^{f\prime}\left(p^f\right) - \overline{S}_2^{f\prime}\left(p^f\right)\right]$ which, by inspection, we may associate with a change in forward market *revenue* rather than a change in the opportunity cost of forward market participation.

On the empirical front, one interesting approach to measuring market power in a dynamic game setting that may prove useful in future work is that of Roeller and Sickles (2000). These authors derive and estimate econometrically a structural model of the European airline industry that posits competition in capacities in the first period, and in prices in the second. They find that firms are significantly less collusive in the two-stage model than under the one-stage specification.[355] They conclude that collusiveness is overestimated whenever competition naturally occurs in two stages. Their model, which links theory to empirical measures of market power, may constitute a promising approach to deriving an empirically-based PCBB for the multi-settlement SFE model.[356] Unlike in Roeller and Sickles' framework, however, intensity of competition in the present SFE-based model cannot be captured by a simple scalar conduct parameter. Even in the case of affine SFs (as for the spot market SFs in the simplified affine example), we require two parameters to specify uniquely a firm's action.

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[355] This observation is consistent with Tirole's (1988, 336) generalization regarding two-period quantity games.

[356] In the present setting, of course, we have focused exclusively on unilateral market power rather than collusion. Also, electricity markets are characterized by repeated competitive interactions of higher frequency than those that plausibly exist in the airline industry. Accordingly, electricity markets are likely to be more sensitive to dynamic effects than is the airline industry.

Prudence and justice tell me that in electricity and steam there is more love for man than in chastity and abstinence from meat.

—Chekhov, Letter to A.S. Suvorin

Appendix A: Proof that firm 1's spot market supply function intersects its residual demand function exactly once

WE BEGIN by arguing that, under our assumptions, firm 1's residual demand function slopes downward. First recall that, at the outset of text section 4.1, we defined firm 1's spot market residual demand function (given arbitrary $\hat{q}_1^f$ and $\hat{q}_2^f$, and for a particular ε^s) as $D^s\left(p^s,\varepsilon^s\right)-\tilde{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$. For convenience, denote this residual demand function as $RD_1^s\left(p^s,\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)$, so that

$$RD_1^s\left(p^s,\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)\equiv D^s\left(p^s,\varepsilon^s\right)-\tilde{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right). \tag{A.1}$$

Since $D^{s'}\left(p^s,\varepsilon^s\right)<0$ (from text subsection 3.1.10) and $\tilde{\Sigma}_2^{s'}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)>0$ (from text subsection 3.1.5) by assumption,[357] we have from eq. (A.1) that

$$RD_1^{s'}\left(p^s,\varepsilon^s;\hat{q}_1^f,\hat{q}_2^f\right)<0 \tag{A.2}$$

for all p^s, and given ε^s, $\hat{q}_1^f$, and $\hat{q}_2^f$.

We now prove that no two realizations of firm 1's residual demand function intersect. Assume, in contradiction, that two arbitrary residual demand functions $RD_1^s\left(p^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$ and $RD_1^s\left(p^s,\hat{\hat{\varepsilon}}^s;\hat{q}_1^f,\hat{q}_2^f\right)$ *do* intersect at a price $p^s=\tilde{p}^s$, where $\hat{\varepsilon}^s\neq\hat{\hat{\varepsilon}}^s$. Algebraically, this assumption is

$$RD_1^s\left(\tilde{p}^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)=RD_1^s\left(\tilde{p}^s,\hat{\hat{\varepsilon}}^s;\hat{q}_1^f,\hat{q}_2^f\right), \tag{A.3}$$

and Figure A.1 below depicts this relationship graphically.

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[357] Where primes ("$\prime$") denote differentiation with respect to p^s.

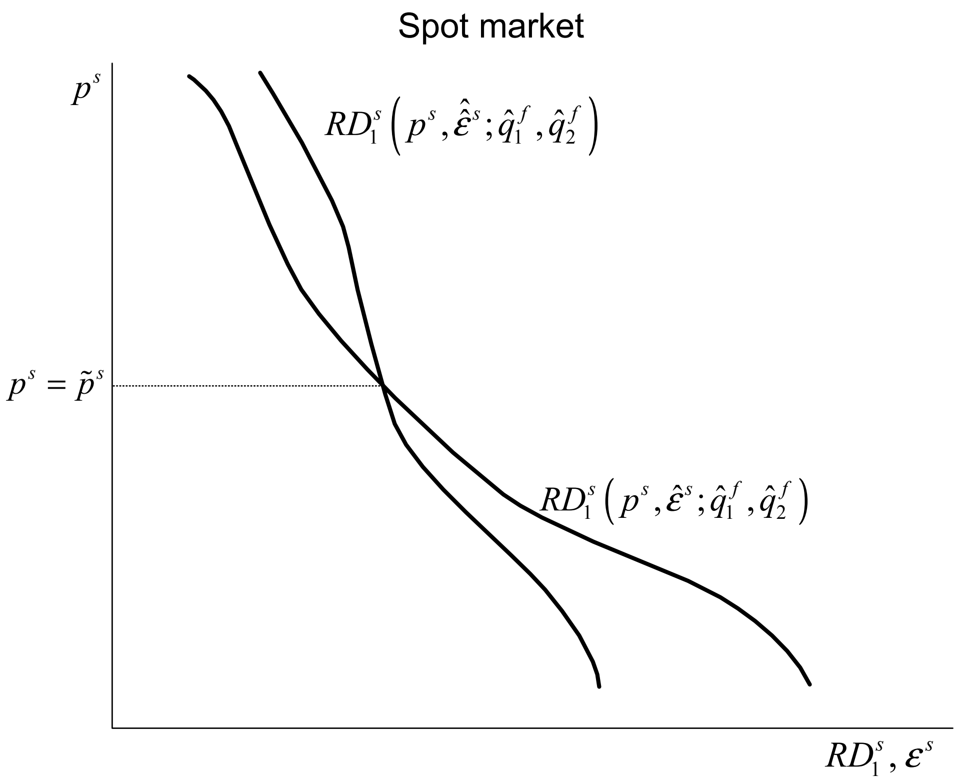

FIGURE A.1: THE FUNCTIONS $RD_1^s\left(\tilde{p}^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$ AND $RD_1^s\left(\tilde{p}^s,\hat{\hat{\varepsilon}}^s;\hat{q}_1^f,\hat{q}_2^f\right)$ INTERSECT AT PRICE $p^s=\tilde{p}^s$ (COUNTERFACTUAL CASE)

Eq. (A.3), however, contradicts our assumption in subsection 3.1.10 of the text that $\partial D^s\left(p^s,\varepsilon^s\right)/\partial\varepsilon^s>0\ \forall p^s$, since at the point of intersection we have $\partial RD_1^s\left(\tilde{p}^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)/\partial\varepsilon^s=0$, and hence, from (A.1), $\partial D^s\left(\tilde{p}^s,\hat{\varepsilon}^s\right)/\partial\varepsilon^s=0$. Thus, it must be that no two realizations of firm 1's residual demand function intersect. ■

Finally, we show that $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ intersects each residual demand function exactly once. Recall by the reasoning in text section 4.1 that, for each ε^s, there exists a unique optimal price p^s. For example, letting $\varepsilon^s=\hat{\varepsilon}^s$, we have that $p^s=p_1^{s*}\left(\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$ for firm 1. For any $\varepsilon^s=\hat{\varepsilon}^s$, there also exists a unique residual demand function, since $\partial D^s\left(p^s,\hat{\varepsilon}^s\right)/\partial\varepsilon^s>0$ in the definition of firm 1's residual demand function, eq. (A.1) above. Therefore, the residual demand function $RD_1^s\left(p^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$ contains—at price $p^s=p_1^{s*}\left(\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$—a unique ex post profit-maximizing point for firm 1.

Figure A.2 below illustrates these relationships. Because we have not yet characterized the properties of the SF $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$, the figure does not depict it, but only indicates its point of intersection with $RD_1^s\left(p^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$.

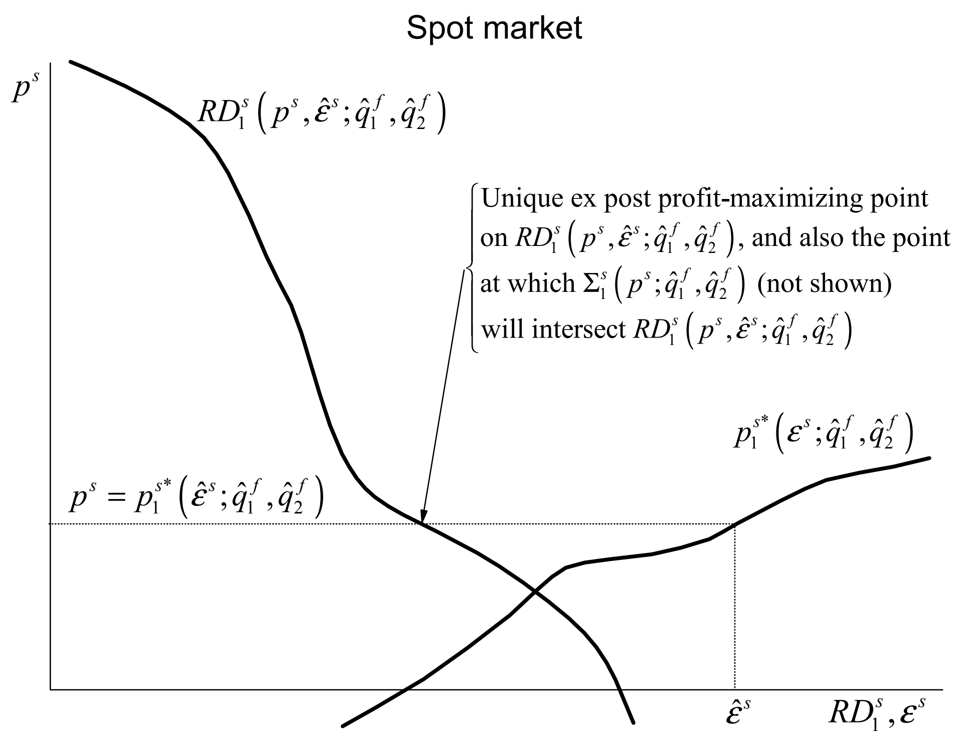

FIGURE A.2: $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ INTERSECTS $RD_1^s\left(p^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$ EXACTLY ONCE

By construction (see text section 4.1), $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ passes through each ex post profit-maximizing point for firm 1 (and only these points). Therefore, $\Sigma_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ will pass through the ex post profit-maximizing point on $RD_1^s\left(p^s,\hat{\varepsilon}^s;\hat{q}_1^f,\hat{q}_2^f\right)$, which lies at price $p^s = p_1^{s*}\left(\hat{\varepsilon}^s; \hat{q}_1^f, \hat{q}_2^f\right)$. Since $p_1^{s*}\left(\hat{\varepsilon}^s; \hat{q}_1^f, \hat{q}_2^f\right)$ is unique, this point of intersection of $\Sigma_1^s\left(p^s; \hat{q}_1^f, \hat{q}_2^f\right)$ and $RD_1^s\left(p^s, \hat{\varepsilon}^s; \hat{q}_1^f, \hat{q}_2^f\right)$ is itself unique. ■

A completely analogous proof applies in the forward market to show that $S_1^f\left(p^f\right)$ intersects firm 1's forward market residual demand function exactly once for each ε_0^f.

Like many businessmen of genius he learned that free competition was wasteful, monopoly efficient. And so he simply set about achieving that efficient monopoly.
—Mario Puzo, *The Godfather*

Appendix B: Second-order sufficient conditions for the optimality of the forward and spot market supply functions

WE FIRST CONSIDER in section B.1 the second-order sufficient conditions for optimality in the spot market, followed in section B.2 by the analogous conditions in the forward market.

B.1 Second-order conditions for the optimality of the spot market SF

The proof in this section parallels that of KM for their Claim 7 (Klemperer and Meyer 1989, 1254). Recall from text eq. (4.3) that the FOC for the provisional spot market problem (assuming an interior solution) is

$$\frac{d\bar{\pi}_1^s\left\{p^s,\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}}{dp^s}=\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]-\hat{q}_1^f$$
$$+\left\{p^s-C_1'\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\right\} \quad \text{(B.1)}$$
$$\cdot\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]$$
$$=0,$$

where the primes " ′" on the spot market demand function and the SFs denote derivatives with respect to p^s. Differentiating eq. (B.1) again to obtain the second-order condition, we have

$$\frac{d^2\bar{\pi}_1^s\left\{p^s,\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}}{\left(dp^s\right)^2}$$
$$=\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]$$
$$+\left\{1-C_1''\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\right\} \quad \text{(B.2)}$$
$$\cdot\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]$$
$$+\left\{p^s-C_1'\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\right\}\left[D^{s\prime\prime}\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^{s\prime\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right].$$

Simplifying and using text eqs. (4.4)–(4.7) to replace $\left[D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]$ with $\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ (where we have also assumed Nash equilibrium between firms 1 and 2), eq. (B.2) becomes

$$\frac{d^2\bar{\pi}_1^s\left\{p^s, \bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right), \hat{q}_1^f, \varepsilon^s\right\}}{\left(dp^s\right)^2}$$
$$\begin{aligned}
&= 2\left[D^{s\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right] \\
&\quad - C_1''\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\left[D^{s\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]^2 \\
&\quad + \left\{p^s - C_1'\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\right\}\left[D^{s\prime\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right].
\end{aligned} \tag{B.3}$$

If $\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ is optimal given $\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$, then it must satisfy eq. (B.1), the optimality condition for firm 1. Again using text eqs. (4.4)–(4.7) to replace $\left[D^s\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]$ by $\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ in eq. (B.1), we get

$$\left\{p^s - C_1'\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\right\}\left[D^{s\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right] = \hat{q}_1^f - \bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right). \tag{B.4}$$

Differentiating both sides of eq. (B.4) with respect to p^s and rearranging, we have

$$\begin{aligned}
&\left\{p^s - C_1'\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\right\}\left[D^{s\prime\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right] \\
&\quad = -\bar{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right) \\
&\qquad - \left\{1 - C_1''\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\bar{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right\}\left[D^{s\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right].
\end{aligned} \tag{B.5}$$

Substituting eq. (B.5) into eq. (B.3) yields

$$\frac{d^2\bar{\pi}_1^s\left\{p^s, \bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right), \hat{q}_1^f, \varepsilon^s\right\}}{\left(dp^s\right)^2}$$
$$\begin{aligned}
&= 2\left[D^{s\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right] \\
&\quad - C_1''\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\left[D^{s\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]^2 - \bar{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right) \\
&\quad - \left\{1 - C_1''\left[\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\bar{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right\}\left[D^{s\prime}\left(p^s,\varepsilon^s\right) - \bar{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right].
\end{aligned}$$

Collecting factors of $\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]$ in the above equation, we have

$$\begin{aligned}&\frac{d^2\overline{\pi}_1^s\left\{p^s,\overline{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}}{\left(dp^s\right)^2}\\&\quad=\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]\cdot\left\{1+C_1''\left[\overline{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\overline{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right\}\\&\quad-C_1''\left[\overline{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]^2-\overline{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right).\end{aligned}\tag{B.6}$$

From eq. (B.6), we may conclude that any SF $\overline{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$ satisfying the spot market FOC (eq. (B.1)) that is also strictly increasing (i.e., $\overline{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)>0$) over its domain is part of an SFE. To see this, note that, given our parametric assumptions and if $\overline{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)>0$, we can sign the terms in eq. (B.6) as

$$\begin{aligned}&\frac{d^2\overline{\pi}_1^s\left\{p^s,\overline{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}}{\left(dp^s\right)^2}\\&\quad=\left[\underbrace{D^{s\prime}\left(p^s,\varepsilon^s\right)}_{-}-\underbrace{\overline{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)}_{+}\right]\cdot\left\{1+\underbrace{C_1''\left[\overline{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]}_{+}\underbrace{\overline{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)}_{+}\right\}\\&\quad-\underbrace{C_1''\left[\overline{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)\right]}_{+}\underbrace{\left[D^{s\prime}\left(p^s,\varepsilon^s\right)-\overline{\Sigma}_2^{s\prime}\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)\right]^2}_{+}-\underbrace{\overline{\Sigma}_1^{s\prime}\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)}_{+}.\end{aligned}$$

Therefore, for this p^s, we have that

$$\frac{d^2\overline{\pi}_1^s\left\{p^s,\overline{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right),\hat{q}_1^f,\varepsilon^s\right\}}{\left(dp^s\right)^2}<0.\tag{B.7}$$

Eq. (B.7) is the second-order sufficient condition for p^s to be a global profit maximum.

B.2 Second-order conditions for the optimality of the forward market SF

The forward market FOC (text eq. (4.19)), is

$$\begin{aligned}\frac{d\tilde{\pi}_1^{tot}\left(p^f,\tilde{S}_2^f\left(p^f\right),\varepsilon_0^f\right)}{dp^f} &= \left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]+p^f\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]\\ &\quad+\mathrm{E}\left[\left.\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right|\varepsilon_0^f\right] \qquad \text{(B.8)}\\ &= 0,\end{aligned}$$

where the primes " ′ " on forward market demand and the SFs denote derivatives with respect to p^f.[358] Differentiating eq. (B.8) with respect to p^f to obtain the second-order condition,[359] we have

$$\begin{aligned}&\frac{d^2\tilde{\pi}_1^{tot}\left(p^f,\tilde{S}_2^f\left(p^f\right),\varepsilon_0^f\right)}{\left(dp^f\right)^2}\\ &= 2\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]+p^f\left[D^{f\prime\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime\prime}\left(p^f\right)\right] \qquad \text{(B.9)}\\ &\quad+\mathrm{E}\left[\left.\frac{d^2\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\left(dp^f\right)^2}\right|\varepsilon_0^f\right].\end{aligned}$$

The first term in eq. (B.9) is negative for strictly increasing $\tilde{S}_2^f\left(p^f\right)$, but without further restrictions on the functional forms of $\bar{\Sigma}_1^s\left(p^s;\hat{q}_1^f,\hat{q}_2^f\right)$, $\bar{\Sigma}_2^s\left(p^s;\hat{q}_2^f,\hat{q}_1^f\right)$, $D^f\left(p^f,\varepsilon_0^f\right)$, and $\tilde{S}_2^f\left(p^f\right)$, the second and third terms in eq. (B.9) are indeterminate in sign.

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[358] This section draws on numerous results from chapters 4 and 6.

[359] And assuming uniform convergence of the expectation integral in this equation.

In this general case, we can make no further progress in signing the terms in eq. (B.9). We must assume that the second and third terms in eq. (B.9) are such that $d^2\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right), \varepsilon_0^f\right) \Big/ \left(dp^f\right)^2 < 0$. Under these assumptions, we conclude that p^f is a global profit maximum for firm 1. In what follows, we (soon) restrict ourselves to the case of the simplified affine example of chapter 5.

To evaluate the derivative inside the expectation on the right-hand side of eq. (B.9), first recall text eq. (4.20):[360]

$$\begin{aligned}
&\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f} \\
&\quad=\frac{\partial\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial q_1^f}\cdot\frac{dq_1^f}{dp^f} \\
&\quad\quad+\frac{\partial\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\partial\tilde{q}_2^f}\cdot\frac{d\tilde{q}_2^f}{dp^f}.
\end{aligned} \tag{B.10}$$

We ultimately want to differentiate eq. (B.10) again with respect to p^f, but first use some results from chapters 4 and 5 to simplify this equation.

Namely, from chapter 4:

- Eqs. (4.25) and (4.26) give expressions for the derivatives of $\bar{\pi}_1^{s*}$ with respect to q_1^f and $\tilde{q}_2^f$;

[360] Recalling text eqs. (3.43) and (3.42), we see that the first and second arguments of $\bar{\pi}_1^{s*}$ are q_1^f and $\tilde{q}_2^f$, respectively. It will be useful shorthand in eq. (B.10) to define derivatives of $\bar{\pi}_1^{s*}$ with respect to these forward market quantities.

- Eqs. (4.31) and (4.32) express dq_1^f / dp^f and $d\tilde{q}_2^f / dp^f$ in terms of forward market SFs.

Making these substitutions in eq. (B.10) yields

$$\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}$$
$$\begin{aligned}&=\left\{-\left[p^s-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\right]\cdot\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial q_1^f}-p^s\right\}\\&\quad\cdot\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]\\&\quad+\left\{-\left[p^s-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\right]\cdot\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial\tilde{q}_2^f}\right\}\cdot\tilde{S}_2^{f\prime}\left(p^f\right),\end{aligned}\tag{B.11}$$

where we recall that, for ease of notation, we introduced $\bar{\Sigma}_2^s\{\cdots\}$ in chapter 4, given by

$$\bar{\Sigma}_2^s\{\cdots\}\equiv\bar{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\}=\bar{\Sigma}_2^s\left\{p^s;\tilde{q}_2^f,q_1^f\right\}. \tag{B.12}$$

Now, we restrict our focus in this discussion to the framework of the simplified affine example of chapter 5. Under the assumption that $\bar{\Sigma}_i^s\left\{p^s;q_i^f,\tilde{q}_j^f\right\}$ $(i,j=1,2;\ i\neq j)$ is affine (recall the Affine Spot Market SFs assumption from section 5.1),[361] we may evaluate the derivatives of $\bar{\Sigma}_2^s\left\{p^s;\tilde{q}_2^f,q_1^f\right\}$ in eq. (B.11) as

$$\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\tilde{q}_2^f,q_1^f\right\}}{\partial q_1^f}=0 \tag{B.13}$$

and

[361] While chapter 5's discussion uses equilibrium quantities $\bar{q}_1^f$ and $\bar{q}_2^f$, we use here the analogous expressions in terms of an arbitrary q_1^f and the imputed quantity for firm 2, $\tilde{q}_2^f$.

$$\frac{\partial \bar{\Sigma}_2^s\left\{p^s;\tilde{q}_2^f,q_1^f\right\}}{\partial \tilde{q}_2^f}=\phi_2\,. \tag{B.14}$$

Substituting eqs. (B.13) and (B.14) into eq. (B.11) yields

$$\begin{aligned}
&\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\\
&\qquad=-p^s\cdot\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]\\
&\qquad\quad-\phi_2\left[p^s-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\right]\tilde{S}_2^{f\prime}\left(p^f\right).
\end{aligned} \tag{B.15}$$

Differentiating eq. (B.15) with respect to p^f, we have

$$\begin{aligned}
&\frac{d^2\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\left(dp^f\right)^2}\\
&\quad=-\frac{dp^s}{dp^f}\cdot\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]-p^s\cdot\left[D^{f\prime\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime\prime}\left(p^f\right)\right]\\
&\quad-\phi_2\left[\frac{dp^s}{dp^f}-C_1''\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\cdot\frac{d}{dp^f}\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\right]\\
&\quad\cdot\tilde{S}_2^{f\prime}\left(p^f\right)-\phi_2\left[p^s-C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\right]\tilde{S}_2^{f\prime\prime}\left(p^f\right).
\end{aligned} \tag{B.16}$$

We now apply several results from chapter 4 to simplify eq. (B.16). Begin by considering the derivative $d\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\big/dp^f$ on the right-hand side of eq. (B.16):

$$\begin{aligned}
&\frac{d}{dp^f}\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)\\
&\qquad=\frac{dD^s\left(p^s,\varepsilon^s\right)}{dp^f}-\frac{d\bar{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\}}{dp^f};
\end{aligned}$$

$$\frac{d}{dp^f}\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)$$

$$= D^{s\prime}\left(p^s,\varepsilon^s\right)\cdot\frac{dp^s}{dp^f}-\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial p^s}\cdot\frac{dp^s}{dp^f}-\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial\tilde{S}_2^f\left(p^f\right)}\cdot\tilde{S}_2^{f\prime}\left(p^f\right) \tag{B.17}$$
$$-\frac{\partial\bar{\Sigma}_2^s\{\cdots\}}{\partial\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]}\cdot\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right].$$

Examining various terms on the right-hand side of eqs. (B.16) and (B.17), we note the following simplifications under the assumptions of chapters 3 and 5:

- $\partial\bar{\Sigma}_2^s\{\cdots\}\big/\partial p^s \equiv \beta_2^s$ given the Affine Spot Market SFs assumption (text eq. (5.2))
- $\partial\bar{\Sigma}_2^s\{\cdots\}\big/\partial\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]=\partial\bar{\Sigma}_2^s\{\cdots\}\big/\partial q_1^f=0$ by differentiating text eq. (5.12), rewritten in terms of $\bar{\Sigma}_2^s\left\{p^s;q_2^f,\tilde{q}_1^f\right\}$
- $\partial\bar{\Sigma}_2^s\{\cdots\}\big/\partial\tilde{S}_2^f\left(p^f\right)=\partial\bar{\Sigma}_2^s\{\cdots\}\big/\partial\tilde{q}_2^f=\phi_2$ by adapting text eq. (5.13) for $i=2$
- $C_1'\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)=C_1'\left(\bar{\Sigma}_1^s\{\cdots\}\right)=C_1'\left(\bar{q}_1^s\right)=c_{01}+c_1\bar{q}_1^s$ recalling the definition of $\bar{\Sigma}_1^s\{\cdots\}$,[362] and the Affine Marginal Production Cost Functions assumption (text eq. (5.1))
- $C_1''\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)=C_1''\left(\bar{\Sigma}_1^s\{\cdots\}\right)=C_1''\left(\bar{q}_1^s\right)=c_1$ by the same reasoning as above

---

[362] Defined symmetrically to $\bar{\Sigma}_2^s\{\cdots\}$ in eq. (B.12).

- From the additive separability of $D^f\left(p^f,\varepsilon_0^f\right)$ in text eq. (3.8), we may write $D^{f\prime}\left(p^f,\varepsilon_0^f\right)\equiv D_0^{f\prime}\left(p^f\right)$

- By similar reasoning as above, we may write $D^{f\prime\prime}\left(p^f,\varepsilon_0^f\right)$ more simply as $D_0^{f\prime\prime}\left(p^f\right)$

- $D^{s\prime}\left(p^s,\varepsilon^s\right)=-\gamma^s$ given the Affine Spot Market Demand Function assumption from section 5.1

Using these results to simplify eq. (B.17), we get

$$\frac{d}{dp^f}\left(D^s\left(p^s,\varepsilon^s\right)-\bar{\Sigma}_2^s\{\cdots\}\right)=-\left(\gamma^s+\beta_2^s\right)\cdot\frac{dp^s}{dp^f}-\phi_2\tilde{S}_2^{f\prime}\left(p^f\right). \tag{B.18}$$

Using the above results along with eq. (B.18) to simplify eq. (B.16) yields

$$\begin{aligned}
&\frac{d^2\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\left(dp^f\right)^2}\\
&\quad=-\frac{dp^s}{dp^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]-p^s\left[D_0^{f\prime\prime}\left(p^f\right)-\tilde{S}_2^{f\prime\prime}\left(p^f\right)\right]\\
&\quad\quad-\phi_2\left\{\frac{dp^s}{dp^f}-c_1\left[-\left(\gamma^s+\beta_2^s\right)\cdot\frac{dp^s}{dp^f}-\phi_2\tilde{S}_2^{f\prime}\left(p^f\right)\right]\right\}\tilde{S}_2^{f\prime}\left(p^f\right)\\
&\quad\quad-\phi_2\left[p^s-\left(c_{01}+c_1\bar{q}_1^s\right)\right]\tilde{S}_2^{f\prime\prime}\left(p^f\right),
\end{aligned}$$

or, rearranging,

$$\frac{d^2\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\left(dp^f\right)^2}$$

$$\begin{aligned}
&=-\frac{dp^s}{dp^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]-p^s\left[D_0^{f\prime\prime}\left(p^f\right)-\tilde{S}_2^{f\prime\prime}\left(p^f\right)\right]\\
&-\frac{dp^s}{dp^f}\cdot\phi_2\left[1+c_1\left(\gamma^s+\beta_2^s\right)\right]\tilde{S}_2^{f\prime}\left(p^f\right)-c_1\phi_2{}^2\left[\tilde{S}_2^{f\prime}\left(p^f\right)\right]^2\\
&-\phi_2\left[p^s-\left(c_{01}+c_1\bar{q}_1^s\right)\right]\tilde{S}_2^{f\prime\prime}\left(p^f\right).
\end{aligned}$$

Recalling text eq. (5.4) for $i=1$, we may substitute $1/\phi_1$ for $\left[1+c_1\left(\gamma^s+\beta_2^s\right)\right]$ in the above equation and collect terms to obtain

$$\frac{d^2\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\left(dp^f\right)^2}$$

$$\begin{aligned}
&=-\frac{dp^s}{dp^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\frac{\phi_1-\phi_2}{\phi_1}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right]-c_1\phi_2{}^2\left[\tilde{S}_2^{f\prime}\left(p^f\right)\right]^2\\
&-p^s\left[D_0^{f\prime\prime}\left(p^f\right)-\tilde{S}_2^{f\prime\prime}\left(p^f\right)\right]-\phi_2\left[p^s-\left(c_{01}+c_1\bar{q}_1^s\right)\right]\tilde{S}_2^{f\prime\prime}\left(p^f\right).
\end{aligned}\tag{B.19}$$

Recall that we defined $\bar{q}_1^s$ as the spot market SF $\bar{\Sigma}_1^s\left\{p^s;q_1^f,\tilde{q}_2^f\right\}$, which, using a variant of text eq. (5.9), we may write as

$$\bar{q}_1^s\equiv\bar{\Sigma}_1^s\left\{p^s;q_1^f,\tilde{q}_2^f\right\}=\left(\phi_1q_1^f-c_{01}\beta_1^s\right)+\beta_1^sp^s.\tag{B.20}$$

Using eq. (B.20), we may simplify further the fourth term on the right-hand side of eq. (B.19). Namely, write the leading factor of this term, $\phi_2\left[p^s-\left(c_{01}+c_1\bar{q}_1^s\right)\right]$, as

$$\phi_2\left[p^s-\left(c_{01}+c_1\bar{q}_1^s\right)\right]=\phi_2\left(p^s-\left\{c_{01}+c_1\left[\left(\phi_1q_1^f-c_{01}\beta_1^s\right)+\beta_1^sp^s\right]\right\}\right),$$

and collect terms in p^s and c_{01} to obtain

$$\phi_2\left[p^s-\left(c_{01}+c_1\overline{q}_1^s\right)\right]=\phi_2\left[\left(1-c_1\beta_1^s\right)p^s-c_{01}\left(1-c_1\beta_1^s\right)-c_1\phi_1 q_1^f\right].$$

Using text eq. (5.4) for $i=1$ and letting $q_1^f \; = D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)$, we may simplify the right-hand side of the above as

$$\phi_2\left[p^s-\left(c_{01}+c_1\overline{q}_1^s\right)\right]=\phi_1\phi_2\left(p^s-\left\{c_{01}+c_1\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\}\right). \qquad \text{(B.21)}$$

We now consolidate the results in this section. Begin by substituting eq. (B.21) into eq. (B.19):

$$\begin{aligned}
&\frac{d^2\overline{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{\left(dp^f\right)^2}\\
&\quad=-\frac{dp^s}{dp^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\frac{\phi_1-\phi_2}{\phi_1}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right]-c_1\phi_2{}^2\left[\tilde{S}_2^{f\prime}\left(p^f\right)\right]^2\\
&\quad\quad-p^s\left[D_0^{f\prime\prime}\left(p^f\right)-\tilde{S}_2^{f\prime\prime}\left(p^f\right)\right]\\
&\quad\quad-\phi_1\phi_2\left(p^s-\left\{c_{01}+c_1\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\}\right)\tilde{S}_2^{f\prime\prime}\left(p^f\right).
\end{aligned}$$

Now substitute this result into the expectation term on the right-hand side of eq. (B.9), using $D_0^{f\prime}\left(p^f\right)$ and $D_0^{f\prime\prime}\left(p^f\right)$ in place of $D^{f\prime}\left(p^f,\varepsilon_0^f\right)$ and $D^{f\prime\prime}\left(p^f,\varepsilon_0^f\right)$, respectively, as before:

$$
\begin{aligned}
&\frac{d^2\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right), \varepsilon_0^f\right)}{\left(dp^f\right)^2} \\
&\quad = 2\left[D_0^{f\prime}\left(p^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] + p^f\left[D_0^{f\prime\prime}\left(p^f\right) - \tilde{S}_2^{f\prime\prime}\left(p^f\right)\right] \\
&\quad - \mathrm{E}\left\{\frac{dp^s}{dp^f}\cdot\left[D_0^{f\prime}\left(p^f\right) - \frac{\phi_1-\phi_2}{\phi_1}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right] + c_1\phi_2{}^2\left[\tilde{S}_2^{f\prime}\left(p^f\right)\right]^2\right. \\
&\qquad + p^s\left[D_0^{f\prime\prime}\left(p^f\right) - \tilde{S}_2^{f\prime\prime}\left(p^f\right)\right] \\
&\qquad \left. + \phi_1\phi_2\left(p^s - \left\{c_{01} + c_1\left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right]\right\}\right)\tilde{S}_2^{f\prime\prime}\left(p^f\right) \middle| p^f\right\}.
\end{aligned}
\tag{B.22}
$$

For an equilibrium $p^f = p^{f*}\left(\varepsilon_0^f\right)$, section 5.4 shows that p^f and ε_0^f are one-to-one. Hence, we may condition in eq. (B.22) on p^f instead of on ε_0^f, as in eq. (B.9). Distributing the expectation operator inside of the braces in the above expression and rearranging, we have

$$
\begin{aligned}
&\frac{d^2\tilde{\pi}_1^{tot}\left(p^f, \tilde{S}_2^f\left(p^f\right), \varepsilon_0^f\right)}{\left(dp^f\right)^2} \\
&\quad = 2\left[D_0^{f\prime}\left(p^f\right) - \tilde{S}_2^{f\prime}\left(p^f\right)\right] - \left[\mathrm{E}\left(p^s \middle| p^f\right) - p^f\right]\left[D_0^{f\prime\prime}\left(p^f\right) - \tilde{S}_2^{f\prime\prime}\left(p^f\right)\right] \\
&\quad - \frac{d\mathrm{E}\left(p^s \middle| p^f\right)}{dp^f}\cdot\left[D_0^{f\prime}\left(p^f\right) - \frac{\phi_1-\phi_2}{\phi_1}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right] - c_1\phi_2{}^2\left[\tilde{S}_2^{f\prime}\left(p^f\right)\right]^2 \\
&\quad - \phi_1\phi_2\left(\mathrm{E}\left(p^s \middle| p^f\right) - \left\{c_{01} + c_1\left[D^f\left(p^f, \varepsilon_0^f\right) - \tilde{S}_2^f\left(p^f\right)\right]\right\}\right)\tilde{S}_2^{f\prime\prime}\left(p^f\right).
\end{aligned}
\tag{B.23}
$$

Consider now the signs of the five terms appearing on the right-hand side of eq. (B.23). Based on various assumptions in the text, we may sign only the first and fourth of these terms for the general forward market problem, as indicated below:

$$
\begin{aligned}
&\frac{d^2\tilde{\pi}_1^{tot}\left(p^f,\tilde{S}_2^f\left(p^f\right),\varepsilon_0^f\right)}{\left(dp^f\right)^2} \\
&\quad = 2\underbrace{\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]}_{-}-\underbrace{\left[\mathrm{E}\left(p^s\middle|p^f\right)-p^f\right]\left[D_0^{f\prime\prime}\left(p^f\right)-\tilde{S}_2^{f\prime\prime}\left(p^f\right)\right]}_{?} \\
&\quad -\underbrace{\frac{d\mathrm{E}\left(p^s\middle|p^f\right)}{dp^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\frac{\phi_1-\phi_2}{\phi_1}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right]}_{?}-\underbrace{c_1\phi_2{}^2\left[\tilde{S}_2^{f\prime}\left(p^f\right)\right]^2}_{+} \\
&\quad -\underbrace{\phi_1\phi_2\left(\mathrm{E}\left(p^s\middle|p^f\right)-\left\{c_{01}+c_1\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]\right\}\right)\tilde{S}_2^{f\prime\prime}\left(p^f\right)}_{?}.
\end{aligned} \tag{B.24}
$$

The three terms marked with "?" on the right-hand side of eq. (B.24) are of indeterminate sign.[363] Because of these analytical indeterminacies, we restrict the quantitative analysis to an imputed admissible forward market SF $\tilde{S}_2^f\left(p^f\right)$ for firm 2 (computed numerically), a domain of forward market prices p^f, and parameter values such that the right-hand side of eq. (B.24) is negative, so that

$$
\frac{d^2\tilde{\pi}_1^{tot}\left(p^f,\tilde{S}_2^f\left(p^f\right),\varepsilon_0^f\right)}{\left(dp^f\right)^2}<0\,. \tag{B.25}
$$

That is, in text chapter 7's specific numerical examples for the forward market problem, we verify numerically for the equilibria we study that the second-order condition expressed by eq. (B.24) and inequality (B.25) in fact holds.

[363] Note that we derived expressions for $D_0^{f\prime}\left(p^f\right)$, $\mathrm{E}\left(p^s\middle|p^f\right)$, and $d\mathrm{E}\left(p^s\middle|p^f\right)\big/dp^f$ in text chapters 6 and 7, and can obtain an expression for $D_0^{f\prime\prime}\left(p^f\right)$ by differentiating $D_0^{f\prime}\left(p^f\right)$. Even if we were to use these expressions to simplify eq. (B.24), however, it would not eliminate the indeterminacy of terms' signs in this equation.

We conclude by noting that, under our assumptions, eq. (B.24) and inequality (B.25) comprise the second-order sufficient condition for p^f to be a global profit maximum for firm 1. We may make a completely symmetric argument for firm 2's global profit maximum.

When you reach an equilibrium in biology you are dead.

—Arnold Mandell

Appendix C: Interpretation of $\psi_1\left(p^f\right)$ and the forward market equilibrium optimality condition

THIS APPENDIX derives eq. (4.43) in the text, rewritten below as eq. (C.1),

$$\psi_1\left(p^f\right) = \mathrm{E}\left(\left.\frac{d\bar{\pi}_1^{s*}\left\{\bar{q}_1^f, \bar{q}_2^f, \varepsilon^s\right\}}{dp^f}\right| p^f\right) + \frac{d\bar{q}_1^f}{dp^f}\cdot \mathrm{E}\left(p^s \middle| p^f\right), \tag{C.1}$$

and provides an interpretation of $\psi_1\left(p^f\right)$ as well as of the forward market equilibrium optimality condition. Eq. (C.1) states that $\psi_1\left(p^f\right)$ is the expected change in firm 1's equilibrium optimal provisional spot profits caused by a marginal change in p^f while netting out the expected change in its forward contract settlement payment, $\left(-p^s q_1^f\right)$, due to this change in p^f. In other words, $\psi_1\left(p^f\right)$ captures the effect of a marginal change in

p^f on firm 1's expectation of spot market revenue less production cost. Later in chapter 8, we also identify $\psi_1\left(p^f\right)$ as firm 1's *strategic effect*, accounting, in part, for the firm's participation in the forward market.

We restate the FOC for the forward market, text eq. (4.19), given the equilibrium imposed for this market later in text chapter 4. As explained in that chapter, this entails

1. replacing $D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)$ pointwise (i.e., for each ε_0^f) with $S_1^f\left(p^f\right)$ (recall text eq. (4.35) and the associated discussion),
2. replacing $D^{f\prime}\left(p^f,\varepsilon_0^f\right)$ with $D_0^{f\prime}\left(p^f\right)$ (using text eq. (3.13)), and
3. conditioning the expectation on p^f rather than on ε_0^f (see note 140).

Making these changes to text eq. (4.19), we may restate this equation as

$$\begin{aligned}\frac{d\tilde{\pi}_1^{tot}\left\{p^f,\tilde{S}_2^f\left(p^f\right),\varepsilon_0^f\right\}}{dp^f} &= S_1^f\left(p^f\right)+p^f\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right] \\ &\quad +\mathrm{E}\left[\left.\frac{d\bar{\pi}_1^{s*}\left\{S_1^f\left(p^f\right),\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right|p^f\right] \qquad \text{(C.2)} \\ &= 0.\end{aligned}$$

Equating eq. (C.2) and text eq. (4.37), a simplified version of the same FOC, we have that

$$
\begin{aligned}
&S_1^f\left(p^f\right)+p^f\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]+\mathrm{E}\left[\left.\frac{d\bar{\pi}_1^{s*}\left\{S_1^f\left(p^f\right),\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right| p^f\right]\\
&=S_1^f\left(p^f\right)+\left[p^f-\mathrm{E}\left(p^s\middle|p^f\right)\right]\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]\\
&\quad-\mathrm{E}\left(\left\{\left[p^s-C_1'\left(\bar{\Sigma}_1^s\left\{p^s;S_1^f\left(p^f\right),\tilde{S}_2^f\left(p^f\right)\right\}\right)\right]\right.\right.\\
&\qquad\cdot\left[\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),S_1^f\left(p^f\right)\right\}}{\partial q_1^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]\right.\\
&\qquad\left.\left.\left.+\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),S_1^f\left(p^f\right)\right\}}{\partial\tilde{q}_2^f}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right]\right\}\middle|p^f\right).
\end{aligned}
\tag{C.3}
$$

Next, recall text eq. (4.39) for $\psi_1\left(p^f\right)$, rewritten below as eq. (C.4):

$$
\begin{aligned}
\psi_1\left(p^f\right)\equiv&-\mathrm{E}\left(\left\{\left[p^s-C_1'\left(\bar{\Sigma}_1^s\left\{p^s;S_1^f\left(p^f\right),\tilde{S}_2^f\left(p^f\right)\right\}\right)\right]\right.\right.\\
&\qquad\cdot\left[\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),S_1^f\left(p^f\right)\right\}}{\partial q_1^f}\cdot\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]\right.\\
&\qquad\left.\left.\left.+\frac{\partial\bar{\Sigma}_2^s\left\{p^s;\tilde{S}_2^f\left(p^f\right),S_1^f\left(p^f\right)\right\}}{\partial\tilde{q}_2^f}\cdot\tilde{S}_2^{f\prime}\left(p^f\right)\right]\right\}\middle|p^f\right).
\end{aligned}
\tag{C.4}
$$

Substituting eq. (C.4) into eq. (C.3), simplifying, and solving for $\psi_1\left(p^f\right)$, we have that

$$
\psi_1\left(p^f\right)=\mathrm{E}\left[\left.\frac{d\bar{\pi}_1^{s*}\left\{S_1^f\left(p^f\right),\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right|p^f\right]+\mathrm{E}\left(p^s\middle|p^f\right)\left[D_0^{f\prime}\left(p^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]. \tag{C.5}
$$

Using text eqs. (3.43) and (3.42) to simplify eq. (C.5) and imposing Nash equilibrium in the forward market, we may rewrite this equation as

$$\psi_1\left(p^f\right)=\mathrm{E}\left(\left.\frac{d\bar{\pi}_1^{s*}\left\{\bar{q}_1^f,\bar{q}_2^f,\varepsilon^s\right\}}{dp^f}\right|p^f\right)+\frac{d\bar{q}_1^f}{dp^f}\cdot\mathrm{E}\left(p^s\middle|p^f\right), \tag{C.6}$$

which is eq. (C.1) (and also text eq. (4.43)), the result that we set out to show.

To conclude this appendix, we provide an interpretation of the forward market equilibrium optimality condition (4.41) in the text. Given an arbitrary forward market demand shock ε_0^f, firm 1 faces the forward market residual demand function $D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)$. Firm 1's forward market revenue $R_1^f\left(p^f,\varepsilon_0^f\right)$ may then be written as

$$R_1^f\left(p^f,\varepsilon_0^f\right)=p^f\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]. \tag{C.7}$$

The derivative of $R_1^f\left(p^f,\varepsilon_0^f\right)$ with respect to p^f is, from eq. (C.7),

$$\frac{\partial R_1^f\left(p^f,\varepsilon_0^f\right)}{\partial p^f}=\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right]+p^f\left[D^{f\prime}\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^{f\prime}\left(p^f\right)\right]. \tag{C.8}$$

Substituting eq. (C.8) into the second equality of text eq. (4.19), firm 1's FOC for the forward market, we get

$$\frac{\partial R_1^f\left(p^f,\varepsilon_0^f\right)}{\partial p^f}+\mathrm{E}\left[\left.\frac{d\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}}{dp^f}\right|\varepsilon_0^f\right]=0\,. \tag{C.9}$$

Eq. (C.9) is a restatement of firm 1's forward market FOC, text eq. (4.19). It is the necessary condition for optimality (assuming interiority) for the problem

$$\begin{aligned}&\tilde{\pi}_1^{tot*}\left(\tilde{S}_2^f(\bullet),\varepsilon_0^f\right)\\&\qquad\equiv\max_{p^f}\left[R_1^f\left(p^f,\varepsilon_0^f\right)+\mathrm{E}\left(\bar{\pi}_1^{s*}\left\{\left[D^f\left(p^f,\varepsilon_0^f\right)-\tilde{S}_2^f\left(p^f\right)\right],\tilde{S}_2^f\left(p^f\right),\varepsilon^s\right\}\middle|\varepsilon_0^f\right)\right],\end{aligned} \tag{C.10}$$

which is itself a restatement of text eq. (4.16), the original forward market problem. Eq. (C.9) indicates that, given firm 2's imputed forward market SF $\tilde{S}_2^f\left(p^f\right)$ and the demand shock ε_0^f, firm 1's optimal price $p^f \equiv p_1^{f*}\left(\varepsilon_0^f\right)$ will be such that the following two marginal changes sum to zero:

1. the marginal change in forward market revenue due to increased p^f, and
2. the marginal change in expected optimal provisional spot market profits (which includes the forward contract settlement payment) due to increased p^f.

> I have yet to see any problem, however complicated, which, when you look at it in the right way, did not become still more complicated.
>
> —Poul Anderson

Appendix D: Computational details of the spot market SFE under the simplified affine example

D.1 Comparative statics of firm *i*'s spot market SF slope β_i^s and parameter ϕ_i with respect to the parameters c_i, c_j, and γ^s

TEXT EQ. (5.4) for the parameter ϕ_i is

$$\phi_i \equiv \frac{1}{1+c_i\left(\gamma^s+\beta_j^s\right)} \quad (i,\, j=1,2;\ i\neq j), \tag{D.1}$$

where under our parametric assumptions, we note (recalling the expression (5.8) in the text) that

$$0<\phi_i<1. \tag{D.2}$$

Rewriting text eq. (5.6) (or (5.11)) for a generic firm *i* and using eq. (D.1) for ϕ_i, we may write the slope of firm *i*'s spot market SF β_i^s as

$$\beta_i^s = \frac{\gamma^s + \beta_j^s}{1 + c_i\left(\gamma^s + \beta_j^s\right)}. \tag{D.3}$$

Interchanging arbitrary subscripts *i* and *j* in eq. (D.3), we may express the corresponding slope β_j^s for firm *j* as

$$\beta_j^s = \frac{\gamma^s + \beta_i^s}{1 + c_j\left(\gamma^s + \beta_i^s\right)}. \tag{D.4}$$

Clearly, we could solve eqs. (D.3) and (D.4) for $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$ explicitly. To sign the derivatives of β_i^s with respect to the parameters c_i, c_j, and γ^s, however, it is simpler to differentiate eqs. (D.3) and (D.4) implicitly, as done in subsections D.1.1–D.1.3 below. Using these results and eq. (D.1), we may similarly sign derivatives of $\phi_i = \phi_i\left(c_i, c_j, \gamma^s\right)$ with respect to these parameters, as in subsections D.1.4–D.1.6. As in the text, we assume throughout this appendix that $\beta_i^s > 0$, $i = 1, 2$.[364]

D.1.1 The partial derivative of $\beta_i^s\left(c_i, c_j, \gamma^s\right)$ with respect to c_i

From eq. (D.3), we may partially differentiate $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$ with respect to c_i as follows:

[364] As noted in text section 1.3, there is a unique solution to eqs. (D.3) and (D.4) for which this is the case. Namely, Rudkevich's (1999) result implies that the solution to eqs. (D.3) and (D.4) for $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$ ($i,\ j = 1, 2;\ i \neq j$) has exactly one root in which both β_1^s and β_2^s are positive.

$$\frac{\partial \beta_i^s}{\partial c_i} = \frac{\partial \beta_i^s\left(c_i, c_j, \gamma^s\right)}{\partial c_i}$$

$$= \frac{\frac{\partial \beta_j^s}{\partial c_i} \cdot \left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right] - \left(\gamma^s + \beta_j^s\right)\left[\left(\gamma^s + \beta_j^s\right) + c_i \cdot \frac{\partial \beta_j^s}{\partial c_i}\right]}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2}$$

$$= \frac{\frac{\partial \beta_j^s}{\partial c_i} - \left(\gamma^s + \beta_j^s\right)^2}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2},$$

which, using the definition (D.1), we may write as

$$\frac{\partial \beta_i^s}{\partial c_i} = \phi_i^2\left[\frac{\partial \beta_j^s}{\partial c_i} - \left(\gamma^s + \beta_j^s\right)^2\right]. \tag{D.5}$$

From eq. (D.4), we may differentiate β_j^s with respect to c_i to obtain

$$\frac{\partial \beta_j^s}{\partial c_i} = \frac{\partial \beta_j^s\left(c_j, c_i, \gamma^s\right)}{\partial c_i}$$

$$= \frac{\frac{\partial \beta_i^s}{\partial c_i} \cdot \left[1 + c_j\left(\gamma^s + \beta_i^s\right)\right] - \left(\gamma^s + \beta_i^s\right) \cdot c_j \cdot \frac{\partial \beta_i^s}{\partial c_i}}{\left[1 + c_j\left(\gamma^s + \beta_i^s\right)\right]^2}$$

$$= \frac{\frac{\partial \beta_i^s}{\partial c_i}}{\left[1 + c_j\left(\gamma^s + \beta_i^s\right)\right]^2},$$

which, again using the definition (D.1), we may write as

$$\frac{\partial \beta_j^s}{\partial c_i} = \phi_j^2 \cdot \frac{\partial \beta_i^s}{\partial c_i}. \tag{D.6}$$

Substituting eq. (D.6) into eq. (D.5) yields

$$\frac{\partial \beta_i^s}{\partial c_i} = \phi_i^2 \left[\phi_j^2 \cdot \frac{\partial \beta_i^s}{\partial c_i} - \left(\gamma^s + \beta_j^s \right)^2 \right]$$

which, solving for $\partial \beta_i^s / \partial c_i$, becomes

$$\frac{\partial \beta_i^s}{\partial c_i} = -\frac{\phi_i^2 \left(\gamma^s + \beta_j^s \right)^2}{1 - \phi_i^2 \phi_j^2}. \tag{D.7}$$

Rewriting text eq. (5.6) for generic firms *i* and *j*, we get

$$\beta_i^s = \phi_i \left(\gamma^s + \beta_j^s \right). \tag{D.8}$$

We may substitute from eq. (D.8) to simplify eq. (D.7) as

$$\frac{\partial \beta_i^s}{\partial c_i} = -\frac{\left(\beta_i^s \right)^2}{1 - \phi_i^2 \phi_j^2}. \tag{D.9}$$

Since the subscript in the expression (D.2) is arbitrary, we have that

$$1 - \phi_i^2 \phi_j^2 > 0. \tag{D.10}$$

Given inequality (D.10), we conclude from eq. (D.9) that

$$\frac{\partial \beta_i^s}{\partial c_i} < 0. \tag{D.11}$$

D.1.2 The partial derivative of $\beta_i^s \left(c_i, c_j, \gamma^s \right)$ with respect to c_j

From eq. (D.6), by symmetry, the partial derivative of $\beta_i^s = \beta_i^s \left(c_i, c_j, \gamma^s \right)$ with respect to c_j is

$$\frac{\partial \beta_i^s}{\partial c_j} = \phi_i^2 \cdot \frac{\partial \beta_j^s}{\partial c_j}. \tag{D.12}$$

Letting $i = j$ in inequality (D.11), we have that $\partial \beta_j^s \big/ \partial c_j < 0$ which, together with eq. (D.12), implies that

$$\frac{\partial \beta_i^s}{\partial c_j} < 0. \tag{D.13}$$

D.1.3 The partial derivative of $\beta_i^s\left(c_i, c_j, \gamma^s\right)$ with respect to γ^s

From eq. (D.3), we may partially differentiate $\beta_i^s = \beta_i^s\left(c_i, c_j, \gamma^s\right)$ with respect to γ^s as follows:

$$\begin{aligned}\frac{\partial \beta_i^s}{\partial \gamma^s} &= \frac{\partial \beta_i^s\left(c_i, c_j, \gamma^s\right)}{\partial \gamma^s} \\ &= \frac{\left(1 + \dfrac{\partial \beta_j^s}{\partial \gamma^s}\right)\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right] - \left(\gamma^s + \beta_j^s\right)\left[c_i\left(1 + \dfrac{\partial \beta_j^s}{\partial \gamma^s}\right)\right]}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2},\end{aligned}$$

which simplifies to

$$\frac{\partial \beta_i^s}{\partial \gamma^s} = \frac{1 + \dfrac{\partial \beta_j^s}{\partial \gamma^s}}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2}.$$

Using the definition (D.1), this equation becomes

$$\frac{\partial \beta_i^s}{\partial \gamma^s} = \phi_i^2\left(1 + \frac{\partial \beta_j^s}{\partial \gamma^s}\right). \tag{D.14}$$

Interchanging arbitrary subscripts *i* and *j* in eq. (D.14), we may write this equation as

$$\frac{\partial \beta_j^s}{\partial \gamma^s} = \phi_j^2 \left(1 + \frac{\partial \beta_i^s}{\partial \gamma^s} \right). \tag{D.15}$$

Substituting eq. (D.15) into eq. (D.14) yields

$$\frac{\partial \beta_i^s}{\partial \gamma^s} = \phi_i^2 \left(1 + \phi_j^2 \left(1 + \frac{\partial \beta_i^s}{\partial \gamma^s} \right) \right),$$

which, solving for $\partial \beta_i^s / \partial \gamma^s$, yields

$$\frac{\partial \beta_i^s}{\partial \gamma^s} = \frac{\phi_i^2 \left(1 + \phi_j^2 \right)}{1 - \phi_i^2 \phi_j^2}. \tag{D.16}$$

Using inequality (D.10), we conclude from eq. (D.16) that

$$\frac{\partial \beta_i^s}{\partial \gamma^s} > 0. \tag{D.17}$$

We collect the signs of the derivatives of β_i^s in inequalities (D.11), (D.13), and (D.17) in the first column of Table 5.1 in the text.

D.1.4 The partial derivative of $\phi_i \left(c_i, c_j, \gamma^s \right)$ with respect to c_i

We may partially differentiate $\phi_i = \phi_i \left(c_i, c_j, \gamma^s \right)$ from eq. (D.1) with respect to c_i as follows:

$$\frac{\partial \phi_i}{\partial c_i} = -\frac{\left(\gamma^s + \beta_j^s \right) + c_i \cdot \dfrac{\partial \beta_j^s}{\partial c_i}}{\left[1 + c_i \left(\gamma^s + \beta_j^s \right) \right]^2},$$

which, using the definition (D.1), becomes

$$\frac{\partial \phi_i}{\partial c_i} = -\phi_i^2 \left[\gamma^s + \beta_j^s + c_i \cdot \frac{\partial \beta_j^s}{\partial c_i} \right]. \tag{D.18}$$

Substituting eq. (D.9) into eq. (D.6) and the result, in turn, into eq. (D.18) for $\partial \beta_j^s \big/ \partial c_i$ yields

$$\frac{\partial \phi_i}{\partial c_i} = -\phi_i^2 \left[\gamma^s + \beta_j^s + c_i \cdot \phi_j^2 \left(-\frac{\left(\beta_i^s\right)^2}{1-\phi_i^2\phi_j^2} \right) \right],$$

which we may write as

$$\frac{\partial \phi_i}{\partial c_i} = -\frac{\phi_i^2\left(\gamma^s + \beta_j^s\right)}{1-\phi_i^2\phi_j^2} \cdot \left[\left(1-\phi_i^2\phi_j^2\right) - \frac{c_i\left(\beta_i^s\right)^2\phi_j^2}{\gamma^s + \beta_j^s} \right]. \tag{D.19}$$

Solving eq. (D.8) for $\beta_i^s \Big/ \left(\gamma^s + \beta_j^s\right) = \phi_i$, we may substitute for this expression in the last term in brackets on the right-hand side of eq. (D.19) to obtain

$$\frac{\partial \phi_i}{\partial c_i} = -\frac{\phi_i^2\left(\gamma^s + \beta_j^s\right)}{1-\phi_i^2\phi_j^2} \cdot \left[\left(1-\phi_i^2\phi_j^2\right) - c_i\beta_i^s\phi_i\phi_j^2 \right]. \tag{D.20}$$

Next, on the right-hand side of eq. (D.20), we (1) substitute for $\phi_i\left(\gamma^s + \beta_j^s\right)$ from eq. (D.8) in the first factor, and (2) rearrange the bracketed expression to obtain

$$\frac{\partial \phi_i}{\partial c_i} = -\frac{\beta_i^s\phi_i}{1-\phi_i^2\phi_j^2} \cdot \left[1-\phi_i\phi_j^2\left(\phi_i + c_i\beta_i^s\right) \right]. \tag{D.21}$$

By rearranging text eq. (5.7), we find that $\phi_i + c_i\beta_i^s = 1$ which, substituted into eq. (D.21), yields simply

$$\frac{\partial \phi_i}{\partial c_i} = -\frac{\beta_i^s \phi_i \left(1 - \phi_i \phi_j^2\right)}{1 - \phi_i^2 \phi_j^2}. \tag{D.22}$$

Since the subscript in the expression (D.2) is arbitrary, we have that $1 - \phi_i \phi_j^2 > 0$ in eq. (D.22). Recalling the inequality (D.10) and our parametric assumptions, both the numerator and the denominator of the ratio on the right-hand side of eq. (D.22) are positive. We conclude from eq. (D.22), therefore, that

$$\frac{\partial \phi_i}{\partial c_i} < 0. \tag{D.23}$$

D.1.5 The partial derivative of $\phi_i\left(c_i, c_j, \gamma^s\right)$ with respect to c_j

We may partially differentiate $\phi_i = \phi_i\left(c_i, c_j, \gamma^s\right)$ from eq. (D.1) with respect to c_j as follows:

$$\frac{\partial \phi_i}{\partial c_j} = -\frac{c_i \cdot \dfrac{\partial \beta_j^s}{\partial c_j}}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2},$$

which, using the definition (D.1), becomes

$$\frac{\partial \phi_i}{\partial c_j} = -c_i \phi_i^2 \cdot \frac{\partial \beta_j^s}{\partial c_j}. \tag{D.24}$$

Interchanging arbitrary subscripts *i* and *j* in eq. (D.9), we may write

$$\frac{\partial \beta_j^s}{\partial c_j} = -\frac{\left(\beta_j^s\right)^2}{1 - \phi_i^2 \phi_j^2}. \tag{D.25}$$

Substituting eq. (D.25) into eq. (D.24) yields

$$\frac{\partial \phi_i}{\partial c_j} = -c_i \phi_i^2 \left(-\frac{\left(\beta_j^s\right)^2}{1-\phi_i^2\phi_j^2} \right),$$

which simplifies to

$$\frac{\partial \phi_i}{\partial c_j} = \frac{c_i \left(\beta_j^s\right)^2 \phi_i^2}{1-\phi_i^2\phi_j^2}. \tag{D.26}$$

Recalling inequality (D.10) and our parametric assumptions, both the numerator and the denominator of the ratio on the right-hand side of eq. (D.26) are positive. We conclude from eq. (D.26), therefore, that

$$\frac{\partial \phi_i}{\partial c_j} > 0. \tag{D.27}$$

D.1.6 The partial derivative of $\phi_i\left(c_i, c_j, \gamma^s\right)$ with respect to γ^s

We may partially differentiate $\phi_i = \phi_i\left(c_i, c_j, \gamma^s\right)$ from eq. (D.1) with respect to γ^s as follows:

$$\frac{\partial \phi_i}{\partial \gamma^s} = -\frac{c_i\left(1+\frac{\partial \beta_j^s}{\partial \gamma^s}\right)}{\left[1+c_i\left(\gamma^s+\beta_j^s\right)\right]^2},$$

which becomes, using the definition (D.1),

$$\frac{\partial \phi_i}{\partial \gamma^s} = -c_i \phi_i^2 \left(1+\frac{\partial \beta_j^s}{\partial \gamma^s}\right). \tag{D.28}$$

Interchanging arbitrary subscripts *i* and *j* in eq. (D.16), we may write

$$\frac{\partial \beta_j^s}{\partial \gamma^s} = \frac{\phi_j^2\left(1+\phi_i^2\right)}{1-\phi_i^2\phi_j^2}. \tag{D.29}$$

Substituting eq. (D.29) into eq. (D.28) yields

$$\frac{\partial \phi_i}{\partial \gamma^s} = -c_i\phi_i^2\left(1+\frac{\phi_j^2\left(1+\phi_i^2\right)}{1-\phi_i^2\phi_j^2}\right),$$

which simplifies to

$$\frac{\partial \phi_i}{\partial \gamma^s} = -\frac{c_i\phi_i^2\left(1+\phi_j^2\right)}{1-\phi_i^2\phi_j^2}. \tag{D.30}$$

Recalling inequality (D.10) and our parametric assumptions, both the numerator and the denominator of the ratio on the right-hand side of eq. (D.30) are positive. We conclude from eq. (D.30), therefore, that

$$\frac{\partial \phi_i}{\partial \gamma^s} < 0. \tag{D.31}$$

We collect the signs of the derivatives of ϕ_i in inequalities (D.23), (D.27), and (D.31) in the second column of Table 5.1 in the text.

D.2 Comparing derivatives of firms' spot market SF slopes β_i^s and β_j^s with respect to the slopes c_i and c_j of a firm's own and the firm's rival's marginal cost function

This section proves inequalities (5.14) and (5.15) in the text, rewritten below as eqs. (D.32) and (D.33) ($i, j = 1, 2;\ i \neq j$):

$$\left|\frac{\partial \beta_i^s}{\partial c_i}\right| > \left|\frac{\partial \beta_i^s}{\partial c_j}\right|, \tag{D.32}$$

and

$$\left|\frac{\partial \beta_i^s}{\partial c_i}\right| > \left|\frac{\partial \beta_j^s}{\partial c_i}\right|, \tag{D.33}$$

whereby inequalities (D.32) and (D.33) obtain at all parameter values consistent with our parametric assumptions.

Equation (D.9) gives an expression for $\partial \beta_i^s / \partial c_i$. To show that inequalities (D.32) and (D.33) hold, we first need to derive expressions for $\partial \beta_i^s / \partial c_j$ and $\partial \beta_j^s / \partial c_i$ from the analysis in section D.1. Substituting eq. (D.25) into eq. (D.12), we may write that

$$\frac{\partial \beta_i^s}{\partial c_j} = \phi_i^2 \left(-\frac{\left(\beta_j^s\right)^2}{1-\phi_i^2 \phi_j^2} \right),$$

or

$$\frac{\partial \beta_i^s}{\partial c_j} = -\frac{\left(\beta_j^s\right)^2 \phi_i^2}{1-\phi_i^2 \phi_j^2}. \tag{D.34}$$

Interchanging arbitrary subscripts *i* and *j* in eq. (D.34) yields

$$\frac{\partial \beta_j^s}{\partial c_i} = -\frac{\left(\beta_i^s\right)^2 \phi_j^2}{1-\phi_i^2 \phi_j^2}. \tag{D.35}$$

Now we may demonstrate that inequality (D.32) holds. Suppose, in contradiction, that it does not, that is,

$$\left|\frac{\partial \beta_i^s}{\partial c_i}\right| \leq \left|\frac{\partial \beta_i^s}{\partial c_j}\right|. \tag{D.36}$$

Substituting eqs. (D.9) and (D.34) into inequality (D.36) yields

$$\left|-\frac{\left(\beta_i^s\right)^2}{1-\phi_i^2\phi_j^2}\right| \leq \left|-\frac{\left(\beta_j^s\right)^2 \phi_i^2}{1-\phi_i^2\phi_j^2}\right|,$$

which we may simplify and rearrange as

$$\left(\frac{\beta_i^s}{\phi_i}\right)^2 \leq \left(\beta_j^s\right)^2. \tag{D.37}$$

Solving eq. (D.8) for the ratio β_i^s / ϕ_i yields

$$\frac{\beta_i^s}{\phi_i} = \gamma^s + \beta_j^s. \tag{D.38}$$

Substituting eq. (D.38) into inequality (D.37), we find that

$$\left(\gamma^s + \beta_j^s\right)^2 \leq \left(\beta_j^s\right)^2,$$

or simplifying,

$$\left(\gamma^s\right)^2 + 2\gamma^s\beta_j^s \leq 0. \tag{D.39}$$

Since inequality (D.39) is false given our parametric assumptions that $\gamma^s > 0$ and $\beta_j^s > 0$, we have a contradiction. Thus, the supposition (D.36) is false and we conclude that inequality (D.32) (identical to inequality (5.14)) holds for our parametric assumptions. ■

We turn next to inequality (D.33), and show that it holds. Suppose, in contradiction, that it does not, that is,

$$\left|\frac{\partial \beta_i^s}{\partial c_i}\right| \le \left|\frac{\partial \beta_j^s}{\partial c_i}\right|. \tag{D.40}$$

Substituting eqs. (D.9) and (D.35) into inequality (D.40) yields

$$\left|-\frac{\left(\beta_i^s\right)^2}{1-\phi_i^2\phi_j^2}\right| \le \left|-\frac{\left(\beta_i^s\right)^2\phi_j^2}{1-\phi_i^2\phi_j^2}\right|,$$

which simplifies to

$$\phi_j^2 \ge 1\,. \tag{D.41}$$

Since inequality (D.41) is false given the expression (D.2), we have a contradiction. Thus, the supposition (D.40) is false and we conclude that inequality (D.33) (identical to inequality (5.15)) also holds under our parametric assumptions. ■

D.3 The geometry of the partial reaction functions $\boldsymbol{\beta_i^s = R_i\left(\beta_j^s\right)}$

In text section 5.3, we defined the *partial reaction functions* $R_i\left(\beta_j^s\right) \equiv \beta_i^s$ in the β_1^s-β_2^s plane. Using eq. (D.3) above for β_i^s, we may write $R_i\left(\beta_j^s\right)$ as

$$\beta_i^s = R_i\left(\beta_j^s\right) \equiv \frac{\gamma^s + \beta_j^s}{1 + c_i\left(\gamma^s + \beta_j^s\right)} \quad (i,\ j = 1,\ 2;\ i \ne j)\,. \tag{D.42}$$

The present section demonstrates that the functions $R_i\left(\beta_j^s\right)$ in eq. (D.42) have the properties claimed in text section 5.3 and depicted in text Figure 5.2.

We first show that each function $R_i\left(\beta_j^s\right)$ is everywhere increasing and concave in its argument $\beta_j^s > 0$. Taking the derivative of $R_i\left(\beta_j^s\right)$ with respect to β_j^s, we get

$$R_i'\left(\beta_j^s\right) = \frac{1 \cdot \left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right] - \left(\gamma^s + \beta_j^s\right) \cdot c_i}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2},$$

which simplifies to

$$R_i'\left(\beta_j^s\right) = \frac{1}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2} > 0. \tag{D.43}$$

From eq. (D.43), the second derivative $R_i''\left(\beta_j^s\right)$ is

$$R_i''\left(\beta_j^s\right) = -\frac{2c_i}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^3} < 0. \tag{D.44}$$

From eqs. (D.43) and (D.44) and given our parametric restrictions, $R_i\left(\beta_j^s\right)$ is everywhere increasing and concave in its argument $\beta_j^s > 0$.

Next, consider how the function $R_i\left(\beta_j^s\right)$ behaves as $\beta_j^s \to 0^+$. From eq. (D.42),

$$\lim_{\beta_j^s \to 0^+} R_i\left(\beta_j^s\right) = \lim_{\beta_j^s \to 0^+} \frac{\gamma^s + \beta_j^s}{1 + c_i\left(\gamma^s + \beta_j^s\right)},$$

or simply

$$\lim_{\beta_j^s \to 0^+} R_i\left(\beta_j^s\right) = \frac{\gamma^s}{1 + c_i\gamma^s} > 0. \tag{D.45}$$

From eq. (D.43),

$$\lim_{\beta_j^s \to 0^+} R_i'\left(\beta_j^s\right) = \lim_{\beta_j^s \to 0^+} \frac{1}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2},$$

which is

$$\lim_{\beta_j^s \to 0^+} R_i'\left(\beta_j^s\right) = \frac{1}{\left(1 + c_i \gamma^s\right)^2} > 0. \tag{D.46}$$

As $\beta_j^s \to 0^+$, eqs. (D.45) and (D.46) indicate that $R_i\left(\beta_j^s\right)$ approaches a positive β_i^s-axis intercept $\gamma^s / \left(1 + c_i \gamma^s\right)$ with a positive slope $1 / \left(1 + c_i \gamma^s\right)^2$.

We now examine the limiting behavior of the function $R_i\left(\beta_j^s\right)$ as $\beta_j^s \to \infty$. From eq. (D.42),

$$\begin{aligned} \lim_{\beta_j^s \to \infty} R_i\left(\beta_j^s\right) &= \lim_{\beta_j^s \to \infty} \frac{\gamma^s + \beta_j^s}{1 + c_i\left(\gamma^s + \beta_j^s\right)} \\ &= \lim_{\beta_j^s \to \infty} \frac{1}{\dfrac{1}{\gamma^s + \beta_j^s} + c_i}, \end{aligned}$$

which is simply

$$\lim_{\beta_j^s \to \infty} R_i\left(\beta_j^s\right) = \frac{1}{c_i} > 0. \tag{D.47}$$

From eq. (D.43),

$$\lim_{\beta_j^s \to \infty} R_i'\left(\beta_j^s\right) = \lim_{\beta_j^s \to \infty} \frac{1}{\left[1 + c_i\left(\gamma^s + \beta_j^s\right)\right]^2},$$

which yields

$$\lim_{\beta_j^s \to \infty} R_i'\left(\beta_j^s\right) = 0\,. \tag{D.48}$$

As $\beta_j^s \to \infty$, eq. (D.47) indicates that $R_i\left(\beta_j^s\right)$ approaches $1/c_i$ (an upper bound, by inequality (D.43)), while by eq. (D.48), the slope $R_i'\left(\beta_j^s\right)$ goes to zero.

Finally, we note that the above properties guarantee a unique intersection of the partial reaction functions $R_1\left(\beta_2^s\right)$ and $R_2\left(\beta_1^s\right)$ in the positive orthant corresponding, naturally, to firms' equilibrium choices of β_1^s and β_2^s.[365] That is, there is a *unique* solution $\left(\beta_1^s, \beta_2^s\right)$ corresponding to a strictly increasing spot market SF for each firm.

[365] See also note 364.

Someone told me that each equation I included in the book would halve the sales.
—Stephen Hawking, *A Brief History of Time*

Appendix E: Computational details of the derivation of optimal forward market supply functions and results of numerical examples

E.1 Supporting analysis for text equations (7.25) and (7.26)

THIS SECTION provides some algebraic details and explicit parameter definitions for the derivation of text eqs. (7.25) and (7.26).

Given the definitions in text section 7.1, we may recast text eqs. (7.11) and (7.12) —the firms' respective forward market equilibrium optimality conditions—as

$$\begin{aligned}
&\left[C_{1,1}^{1}\bar{S}_{1}^{f}\left(p^{f}\right)+C_{1,2}^{1}\bar{S}_{2}^{f}\left(p^{f}\right)+C_{1,3}^{1}p^{f}+C_{1,4}^{1}\right]\bar{S}_{1}^{f\prime}\left(p^{f}\right)\\
&\quad+\left[C_{2,1}^{1}\bar{S}_{1}^{f}\left(p^{f}\right)+C_{2,2}^{1}\bar{S}_{2}^{f}\left(p^{f}\right)+C_{2,3}^{1}p^{f}+C_{2,4}^{1}\right]\bar{S}_{2}^{f\prime}\left(p^{f}\right)\\
&\quad+\left[C_{3,1}^{1}\bar{S}_{1}^{f}\left(p^{f}\right)+C_{3,2}^{1}\bar{S}_{2}^{f}\left(p^{f}\right)+C_{3,3}^{1}p^{f}+C_{3,4}^{1}\right]=0
\end{aligned} \tag{E.1}$$

and

$$
\begin{aligned}
&\left[C_{1,1}^2\bar{S}_1^f\left(p^f\right)+C_{1,2}^2\bar{S}_2^f\left(p^f\right)+C_{1,3}^2p^f+C_{1,4}^2\right]\bar{S}_1^{f\prime}\left(p^f\right)\\
&\qquad+\left[C_{2,1}^2\bar{S}_1^f\left(p^f\right)+C_{2,2}^2\bar{S}_2^f\left(p^f\right)+C_{2,3}^2p^f+C_{2,4}^2\right]\bar{S}_2^{f\prime}\left(p^f\right)\\
&\qquad+\left[C_{3,1}^2\bar{S}_1^f\left(p^f\right)+C_{3,2}^2\bar{S}_2^f\left(p^f\right)+C_{3,3}^2p^f+C_{3,4}^2\right]=0.
\end{aligned}
\tag{E.2}
$$

We may define the coefficients $C_{k,l}^i$ in eqs. (E.1) and (E.2) by direct comparison with text eqs. (7.11) and (7.12) as follows:

$$
\left.
\begin{aligned}
&C_{1,1}^1=\phi_1\omega_a\left(1-\phi_1\right)\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right],\\
&C_{1,2}^1=\phi_1\omega_a\left(1-\phi_2\right)\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right],\\
&C_{1,3}^1=-\phi_1\left(2-\gamma^s\omega_a\right)\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right],\\
&C_{1,4}^1=\phi_1\omega_a\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\left[\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{v}_R-\frac{\sigma_{v_R^2,v_R}}{2\sigma_{v_R}^2}\right)\right],\\
&C_{2,1}^1=c_1\phi_1\phi_2\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\\
&\qquad+\omega_a\left(1-\phi_1\right)\left\{\lambda_R\sigma_{v_R}^2\omega_a\left[\left(1-\phi_1\phi_2\right)+\phi_2\left(1-\gamma^s\omega_a\right)\right]+\phi_2\right\},\\
&C_{2,2}^1=\omega_a\left(1-\phi_2\right)\left\{\lambda_R\sigma_{v_R}^2\omega_a\left[\left(1-\phi_1\phi_2\right)+\phi_2\left(1-\gamma^s\omega_a\right)\right]+\phi_2\right\},\\
&C_{2,3}^1=-\left(\left(2-\gamma^s\omega_a\right)\left\{\lambda_R\sigma_{v_R}^2\omega_a\left[1+\phi_2\left(1-\gamma^s\omega_a\right)\right]+\phi_2\right\}+\phi_1\phi_2\right),\\
&C_{2,4}^1=c_{01}\phi_1\phi_2\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\\
&\qquad+\omega_a\left[\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{v}_R-\frac{\sigma_{v_R^2,v_R}}{2\sigma_{v_R}^2}\right)\right]\left\{\lambda_R\sigma_{v_R}^2\omega_a\left[\left(1-\phi_1\phi_2\right)+\phi_2\left(1-\gamma^s\omega_a\right)\right]+\phi_2\right\},\\
&C_{3,1}^1=\left(1-\phi_1\right)+\left[1+\lambda_R\sigma_{v_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right],\\
&C_{3,2}^1=1-\phi_2,\\
&C_{3,3}^1=-\frac{2-\gamma^s\omega_a}{\omega_a},\\
&C_{3,4}^1=\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{v}_R-\frac{\sigma_{v_R^2,v_R}}{2\sigma_{v_R}^2}\right),
\end{aligned}
\right\}
\tag{E.3}
$$

and

$$\left.\begin{aligned}
C_{1,1}^2 &= \omega_a\left(1-\phi_1\right)\left\{\lambda_R\sigma_{\nu_R}^2\omega_a\left[\left(1-\phi_1\phi_2\right)+\phi_1\left(1-\gamma^s\omega_a\right)\right]+\phi_1\right\},\\
C_{1,2}^2 &= c_2\phi_1\phi_2\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\\
&\quad+\omega_a\left(1-\phi_2\right)\left\{\lambda_R\sigma_{\nu_R}^2\omega_a\left[\left(1-\phi_1\phi_2\right)+\phi_1\left(1-\gamma^s\omega_a\right)\right]+\phi_1\right\},\\
C_{1,3}^2 &= -\left(\left(2-\gamma^s\omega_a\right)\left\{\lambda_R\sigma_{\nu_R}^2\omega_a\left[1+\phi_1\left(1-\gamma^s\omega_a\right)\right]+\phi_1\right\}+\phi_1\phi_2\right),\\
C_{1,4}^2 &= c_{02}\phi_1\phi_2\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right]\\
&\quad+\omega_a\left[\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{\nu}_R-\frac{\sigma_{\nu_R^2,\nu_R}}{2\sigma_{\nu_R}^2}\right)\right]\left\{\lambda_R\sigma_{\nu_R}^2\omega_a\left[\left(1-\phi_1\phi_2\right)+\phi_1\left(1-\gamma^s\omega_a\right)\right]+\phi_1\right\},\\
C_{2,1}^2 &= \phi_2\omega_a\left(1-\phi_1\right)\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right],\\
C_{2,2}^2 &= \phi_2\omega_a\left(1-\phi_2\right)\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right],\\
C_{2,3}^2 &= -\phi_2\left(2-\gamma^s\omega_a\right)\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right],\\
C_{2,4}^2 &= \phi_2\omega_a\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(1-\gamma^s\omega_a\right)\right]\left[\omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{\nu}_R-\frac{\sigma_{\nu_R^2,\nu_R}}{2\sigma_{\nu_R}^2}\right)\right],\\
C_{3,1}^2 &= 1-\phi_1,\\
C_{3,2}^2 &= \left(1-\phi_2\right)+\left[1+\lambda_R\sigma_{\nu_R}^2\omega_a\left(2-\gamma^s\omega_a\right)\right],\\
C_{3,3}^2 &= -\frac{2-\gamma^s\omega_a}{\omega_a},\\
C_{3,4}^2 &= \omega_b+\left(2-\gamma^s\omega_a\right)\left(\bar{\nu}_R-\frac{\sigma_{\nu_R^2,\nu_R}}{2\sigma_{\nu_R}^2}\right).
\end{aligned}\right\} \tag{E.4}$$

We next show that the coefficients of the form $\mathscr{P}_j^1\mathscr{P}_k^2-\mathscr{P}_k^1\mathscr{P}_j^2$ in text eqs. (7.20) and (7.21) are *quadratic forms* in the elements of $\bar{S}^{f++}\left(p^f\right)$, where

$$\bar{S}^{f++}\left(p^f\right)\equiv\begin{pmatrix}\bar{S}_1^f\left(p^f\right) & \bar{S}_2^f\left(p^f\right) & p^f & 1\end{pmatrix}^{\top} \tag{E.5}$$

from text eq. (7.13). Using the definition of $\mathscr{P}_k^i$ from text eq. (7.17), we may express the coefficients in text eqs. (7.20) and (7.21) in terms of the coefficients $C_{k,l}^i$ above as follows. In text eq. (7.20), $\mathscr{P}_1^1\mathscr{P}_2^2 - \mathscr{P}_2^1\mathscr{P}_1^2$ is

$$\begin{aligned} \mathscr{P}_1^1\mathscr{P}_2^2 - \mathscr{P}_2^1\mathscr{P}_1^2 = &\left[C_{1,1}^1 \bar{S}_1^f\left(p^f\right) + C_{1,2}^1 \bar{S}_2^f\left(p^f\right) + C_{1,3}^1 p^f + C_{1,4}^1 \right] \\ &\cdot \left[C_{2,1}^2 \bar{S}_1^f\left(p^f\right) + C_{2,2}^2 \bar{S}_2^f\left(p^f\right) + C_{2,3}^2 p^f + C_{2,4}^2 \right] \\ &- \left[C_{2,1}^1 \bar{S}_1^f\left(p^f\right) + C_{2,2}^1 \bar{S}_2^f\left(p^f\right) + C_{2,3}^1 p^f + C_{2,4}^1 \right] \\ &\cdot \left[C_{1,1}^2 \bar{S}_1^f\left(p^f\right) + C_{1,2}^2 \bar{S}_2^f\left(p^f\right) + C_{1,3}^2 p^f + C_{1,4}^2 \right], \end{aligned} \tag{E.6}$$

or expanding the products,

$$\begin{aligned} \mathscr{P}_1^1\mathscr{P}_2^2 - \mathscr{P}_2^1\mathscr{P}_1^2 = &\left(C_{1,1}^1 C_{2,1}^2 - C_{2,1}^1 C_{1,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)\right]^2 \\ &+ \left(C_{1,2}^1 C_{2,2}^2 - C_{2,2}^1 C_{1,2}^2\right)\left[\bar{S}_2^f\left(p^f\right)\right]^2 \\ &+ \left(C_{1,3}^1 C_{2,3}^2 - C_{2,3}^1 C_{1,3}^2\right)\left[p^f\right]^2 \\ &+ \left(C_{1,1}^1 C_{2,2}^2 + C_{1,2}^1 C_{2,1}^2 - C_{2,1}^1 C_{1,2}^2 - C_{2,2}^1 C_{1,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)\bar{S}_2^f\left(p^f\right)\right] \\ &+ \left(C_{1,1}^1 C_{2,3}^2 + C_{1,3}^1 C_{2,1}^2 - C_{2,1}^1 C_{1,3}^2 - C_{2,3}^1 C_{1,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)p^f\right] \\ &+ \left(C_{1,2}^1 C_{2,3}^2 + C_{1,3}^1 C_{2,2}^2 - C_{2,2}^1 C_{1,3}^2 - C_{2,3}^1 C_{1,2}^2\right)\left[\bar{S}_2^f\left(p^f\right)p^f\right] \\ &+ \left(C_{1,1}^1 C_{2,4}^2 + C_{1,4}^1 C_{2,1}^2 - C_{2,1}^1 C_{1,4}^2 - C_{2,4}^1 C_{1,1}^2\right)\bar{S}_1^f\left(p^f\right) \\ &+ \left(C_{1,2}^1 C_{2,4}^2 + C_{1,4}^1 C_{2,2}^2 - C_{2,2}^1 C_{1,4}^2 - C_{2,4}^1 C_{1,2}^2\right)\bar{S}_2^f\left(p^f\right) \\ &+ \left(C_{1,3}^1 C_{2,4}^2 + C_{1,4}^1 C_{2,3}^2 - C_{2,3}^1 C_{1,4}^2 - C_{2,4}^1 C_{1,3}^2\right)p^f \\ &+ \left(C_{1,4}^1 C_{2,4}^2 - C_{2,4}^1 C_{1,4}^2\right). \end{aligned} \tag{E.7}$$

In text eq. (7.21), $\mathscr{P}_2^1\mathscr{P}_1^2 - \mathscr{P}_1^1\mathscr{P}_2^2$ is just the additive inverse of $\mathscr{P}_1^1\mathscr{P}_2^2 - \mathscr{P}_2^1\mathscr{P}_1^2$ given by eq. (E.6), so we have from eq. (E.7) that

$$
\begin{aligned}
\mathscr{P}_2^1\mathscr{P}_1^2 - \mathscr{P}_1^1\mathscr{P}_2^2 = {} & \left(C_{2,1}^1 C_{1,1}^2 - C_{1,1}^1 C_{2,1}^2\right)\left[\overline{S}_1^f\left(p^f\right)\right]^2 \\
& + \left(C_{2,2}^1 C_{1,2}^2 - C_{1,2}^1 C_{2,2}^2\right)\left[\overline{S}_2^f\left(p^f\right)\right]^2 \\
& + \left(C_{2,3}^1 C_{1,3}^2 - C_{1,3}^1 C_{2,3}^2\right)\left[p^f\right]^2 \\
& + \left(C_{2,1}^1 C_{1,2}^2 + C_{2,2}^1 C_{1,1}^2 - C_{1,1}^1 C_{2,2}^2 - C_{1,2}^1 C_{2,1}^2\right)\left[\overline{S}_1^f\left(p^f\right)\overline{S}_2^f\left(p^f\right)\right] \\
& + \left(C_{2,1}^1 C_{1,3}^2 + C_{2,3}^1 C_{1,1}^2 - C_{1,1}^1 C_{2,3}^2 - C_{1,3}^1 C_{2,1}^2\right)\left[\overline{S}_1^f\left(p^f\right)p^f\right] \\
& + \left(C_{2,2}^1 C_{1,3}^2 + C_{2,3}^1 C_{1,2}^2 - C_{1,2}^1 C_{2,3}^2 - C_{1,3}^1 C_{2,2}^2\right)\left[\overline{S}_2^f\left(p^f\right)p^f\right] \\
& + \left(C_{2,1}^1 C_{1,4}^2 + C_{2,4}^1 C_{1,1}^2 - C_{1,1}^1 C_{2,4}^2 - C_{1,4}^1 C_{2,1}^2\right)\overline{S}_1^f\left(p^f\right) \\
& + \left(C_{2,2}^1 C_{1,4}^2 + C_{2,4}^1 C_{1,2}^2 - C_{1,2}^1 C_{2,4}^2 - C_{1,4}^1 C_{2,2}^2\right)\overline{S}_2^f\left(p^f\right) \\
& + \left(C_{2,3}^1 C_{1,4}^2 + C_{2,4}^1 C_{1,3}^2 - C_{1,3}^1 C_{2,4}^2 - C_{1,4}^1 C_{2,3}^2\right)p^f \\
& + \left(C_{2,4}^1 C_{1,4}^2 - C_{1,4}^1 C_{2,4}^2\right).
\end{aligned}
\tag{E.8}
$$

In text eq. (7.20), $\mathscr{P}_2^1\mathscr{P}_3^2 - \mathscr{P}_3^1\mathscr{P}_2^2$ is

$$
\begin{aligned}
\mathscr{P}_2^1\mathscr{P}_3^2 - \mathscr{P}_3^1\mathscr{P}_2^2 = {} & \left[C_{2,1}^1\overline{S}_1^f\left(p^f\right) + C_{2,2}^1\overline{S}_2^f\left(p^f\right) + C_{2,3}^1 p^f + C_{2,4}^1\right] \\
& \cdot\left[C_{3,1}^2\overline{S}_1^f\left(p^f\right) + C_{3,2}^2\overline{S}_2^f\left(p^f\right) + C_{3,3}^2 p^f + C_{3,4}^2\right] \\
& - \left[C_{3,1}^1\overline{S}_1^f\left(p^f\right) + C_{3,2}^1\overline{S}_2^f\left(p^f\right) + C_{3,3}^1 p^f + C_{3,4}^1\right] \\
& \cdot\left[C_{2,1}^2\overline{S}_1^f\left(p^f\right) + C_{2,2}^2\overline{S}_2^f\left(p^f\right) + C_{2,3}^2 p^f + C_{2,4}^2\right],
\end{aligned}
\tag{E.9}
$$

or expanding the products,

$$
\begin{aligned}
\mathscr{P}_2^1\mathscr{P}_3^2 - \mathscr{P}_3^1\mathscr{P}_2^2 = &\left(C_{2,1}^1C_{3,1}^2 - C_{3,1}^1C_{2,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)\right]^2 \\
&+\left(C_{2,2}^1C_{3,2}^2 - C_{3,2}^1C_{2,2}^2\right)\left[\bar{S}_2^f\left(p^f\right)\right]^2 \\
&+\left(C_{2,3}^1C_{3,3}^2 - C_{3,3}^1C_{2,3}^2\right)\left[p^f\right]^2 \\
&+\left(C_{2,1}^1C_{3,2}^2 + C_{2,2}^1C_{3,1}^2 - C_{3,1}^1C_{2,2}^2 - C_{3,2}^1C_{2,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)\bar{S}_2^f\left(p^f\right)\right] \\
&+\left(C_{2,1}^1C_{3,3}^2 + C_{2,3}^1C_{3,1}^2 - C_{3,1}^1C_{2,3}^2 - C_{3,3}^1C_{2,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)p^f\right] \\
&+\left(C_{2,2}^1C_{3,3}^2 + C_{2,3}^1C_{3,2}^2 - C_{3,2}^1C_{2,3}^2 - C_{3,3}^1C_{2,2}^2\right)\left[\bar{S}_2^f\left(p^f\right)p^f\right] \\
&+\left(C_{2,1}^1C_{3,4}^2 + C_{2,4}^1C_{3,1}^2 - C_{3,1}^1C_{2,4}^2 - C_{3,4}^1C_{2,1}^2\right)\bar{S}_1^f\left(p^f\right) \\
&+\left(C_{2,2}^1C_{3,4}^2 + C_{2,4}^1C_{3,2}^2 - C_{3,2}^1C_{2,4}^2 - C_{3,4}^1C_{2,2}^2\right)\bar{S}_2^f\left(p^f\right) \\
&+\left(C_{2,3}^1C_{3,4}^2 + C_{2,4}^1C_{3,3}^2 - C_{3,3}^1C_{2,4}^2 - C_{3,4}^1C_{2,3}^2\right)p^f \\
&+\left(C_{2,4}^1C_{3,4}^2 - C_{3,4}^1C_{2,4}^2\right).
\end{aligned}
\tag{E.10}
$$

In text eq. (7.21), $\mathscr{P}_1^1\mathscr{P}_3^2 - \mathscr{P}_3^1\mathscr{P}_1^2$ is

$$
\begin{aligned}
\mathscr{P}_1^1\mathscr{P}_3^2 - \mathscr{P}_3^1\mathscr{P}_1^2 = &\left[C_{1,1}^1\bar{S}_1^f\left(p^f\right) + C_{1,2}^1\bar{S}_2^f\left(p^f\right) + C_{1,3}^1p^f + C_{1,4}^1\right] \\
&\cdot\left[C_{3,1}^2\bar{S}_1^f\left(p^f\right) + C_{3,2}^2\bar{S}_2^f\left(p^f\right) + C_{3,3}^2p^f + C_{3,4}^2\right] \\
&-\left[C_{3,1}^1\bar{S}_1^f\left(p^f\right) + C_{3,2}^1\bar{S}_2^f\left(p^f\right) + C_{3,3}^1p^f + C_{3,4}^1\right] \\
&\cdot\left[C_{1,1}^2\bar{S}_1^f\left(p^f\right) + C_{1,2}^2\bar{S}_2^f\left(p^f\right) + C_{1,3}^2p^f + C_{1,4}^2\right],
\end{aligned}
\tag{E.11}
$$

or expanding the products,

$$
\begin{aligned}
\mathcal{P}_1^1\mathcal{P}_3^2 - \mathcal{P}_3^1\mathcal{P}_1^2 = &\left(C_{1,1}^1 C_{3,1}^2 - C_{3,1}^1 C_{1,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)\right]^2 \\
&+\left(C_{1,2}^1 C_{3,2}^2 - C_{3,2}^1 C_{1,2}^2\right)\left[\bar{S}_2^f\left(p^f\right)\right]^2 \\
&+\left(C_{1,3}^1 C_{3,3}^2 - C_{3,3}^1 C_{1,3}^2\right)\left[p^f\right]^2 \\
&+\left(C_{1,1}^1 C_{3,2}^2 + C_{1,2}^1 C_{3,1}^2 - C_{3,1}^1 C_{1,2}^2 - C_{3,2}^1 C_{1,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)\bar{S}_2^f\left(p^f\right)\right] \\
&+\left(C_{1,1}^1 C_{3,3}^2 + C_{1,3}^1 C_{3,1}^2 - C_{3,1}^1 C_{1,3}^2 - C_{3,3}^1 C_{1,1}^2\right)\left[\bar{S}_1^f\left(p^f\right)p^f\right] \\
&+\left(C_{1,2}^1 C_{3,3}^2 + C_{1,3}^1 C_{3,2}^2 - C_{3,2}^1 C_{1,3}^2 - C_{3,3}^1 C_{1,2}^2\right)\left[\bar{S}_2^f\left(p^f\right)p^f\right] \\
&+\left(C_{1,1}^1 C_{3,4}^2 + C_{1,4}^1 C_{3,1}^2 - C_{3,1}^1 C_{1,4}^2 - C_{3,4}^1 C_{1,1}^2\right)\bar{S}_1^f\left(p^f\right) \\
&+\left(C_{1,2}^1 C_{3,4}^2 + C_{1,4}^1 C_{3,2}^2 - C_{3,2}^1 C_{1,4}^2 - C_{3,4}^1 C_{1,2}^2\right)\bar{S}_2^f\left(p^f\right) \\
&+\left(C_{1,3}^1 C_{3,4}^2 + C_{1,4}^1 C_{3,3}^2 - C_{3,3}^1 C_{1,4}^2 - C_{3,4}^1 C_{1,3}^2\right)p^f \\
&+\left(C_{1,4}^1 C_{3,4}^2 - C_{3,4}^1 C_{1,4}^2\right).
\end{aligned}
\tag{E.12}
$$

The right-hand sides of each of eqs. (E.7), (E.8), (E.10), and (E.12) are indeed *quadratic forms* in the elements of $\bar{S}^{f++}\left(p^f\right)$. Below, we specify explicitly the $\left(n+2\right)$ x $\left(n+2\right)$ symmetric coefficient matrices $\mathcal{Q}_{jk}$ associated with these four quadratic forms. Text eq. (7.23), rewritten below as eq. (E.13),

$$
\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{jk}\bar{S}^{f++}\left(p^f\right) \equiv \mathcal{P}_j^1\mathcal{P}_k^2 - \mathcal{P}_k^1\mathcal{P}_j^2 , \tag{E.13}
$$

defined implicitly the coefficient matrix $\mathcal{Q}_{jk}$ with reference to the quadratic forms $\mathcal{P}_j^1\mathcal{P}_k^2 - \mathcal{P}_k^1\mathcal{P}_j^2$ (from the left-hand sides of eqs. (E.7), (E.8), (E.10), and (E.12)). Recalling the four quadratic forms in text eqs. (7.25) and (7.26) of the form $\bar{S}^{f++}\left(p^f\right)^\top \mathcal{Q}_{jk}\bar{S}^{f++}\left(p^f\right)$, each has a corresponding coefficient matrix $\mathcal{Q}_{12}$, $\mathcal{Q}_{21}$, $\mathcal{Q}_{23}$, and $\mathcal{Q}_{13}$ which we now specify.

Denote the element in row *x* and column *y* of the coefficient matrix $\mathcal{Q}_{jk}$ as q_{jk}^{xy}. In terms of the coefficients $C_{k,l}^{i}$ given in the expressions (E.3) and (E.4), the elements of $\mathcal{Q}_{12}$ are as follows (see the right-hand side of eq. (E.7)):

$$\left.\begin{array}{ll}
\begin{array}{c}\text{Diagonal elements}\\ \text{of } \mathcal{Q}_{12}\end{array} & \begin{cases}
q_{12}^{11} = C_{1,1}^{1}C_{2,1}^{2} - C_{2,1}^{1}C_{1,1}^{2} \\
q_{12}^{22} = C_{1,2}^{1}C_{2,2}^{2} - C_{2,2}^{1}C_{1,2}^{2} \\
q_{12}^{33} = C_{1,3}^{1}C_{2,3}^{2} - C_{2,3}^{1}C_{1,3}^{2} \\
q_{12}^{44} = C_{1,4}^{1}C_{2,4}^{2} - C_{2,4}^{1}C_{1,4}^{2}
\end{cases} \\
\begin{array}{c}\text{Off-diagonal elements}\\ \text{of } \mathcal{Q}_{12}\end{array} & \begin{cases}
q_{12}^{12} = q_{12}^{21} = \dfrac{1}{2}\left(C_{1,1}^{1}C_{2,2}^{2} + C_{1,2}^{1}C_{2,1}^{2} - C_{2,1}^{1}C_{1,2}^{2} - C_{2,2}^{1}C_{1,1}^{2}\right) \\
q_{12}^{13} = q_{12}^{31} = \dfrac{1}{2}\left(C_{1,1}^{1}C_{2,3}^{2} + C_{1,3}^{1}C_{2,1}^{2} - C_{2,1}^{1}C_{1,3}^{2} - C_{2,3}^{1}C_{1,1}^{2}\right) \\
q_{12}^{14} = q_{12}^{41} = \dfrac{1}{2}\left(C_{1,1}^{1}C_{2,4}^{2} + C_{1,4}^{1}C_{2,1}^{2} - C_{2,1}^{1}C_{1,4}^{2} - C_{2,4}^{1}C_{1,1}^{2}\right) \\
q_{12}^{23} = q_{12}^{32} = \dfrac{1}{2}\left(C_{1,2}^{1}C_{2,3}^{2} + C_{1,3}^{1}C_{2,2}^{2} - C_{2,2}^{1}C_{1,3}^{2} - C_{2,3}^{1}C_{1,2}^{2}\right) \\
q_{12}^{24} = q_{12}^{42} = \dfrac{1}{2}\left(C_{1,2}^{1}C_{2,4}^{2} + C_{1,4}^{1}C_{2,2}^{2} - C_{2,2}^{1}C_{1,4}^{2} - C_{2,4}^{1}C_{1,2}^{2}\right) \\
q_{12}^{34} = q_{12}^{43} = \dfrac{1}{2}\left(C_{1,3}^{1}C_{2,4}^{2} + C_{1,4}^{1}C_{2,3}^{2} - C_{2,3}^{1}C_{1,4}^{2} - C_{2,4}^{1}C_{1,3}^{2}\right).
\end{cases}
\end{array}\right\} \quad \text{(E.14)}$$

Since $\mathcal{Q}_{21} = -\mathcal{Q}_{12}$, the elements of $\mathcal{Q}_{21}$ are simply the additive inverses of the expressions in (E.14) above (see the right-hand side of eq. (E.8)). Similar to the above, in terms of the coefficients $C_{k,l}^{i}$ given in the expressions (E.3) and (E.4) we write the elements of $\mathcal{Q}_{21}$ as follows:

$$
\left.\begin{array}{ll}
\begin{array}{c}\text{Diagonal elements}\\ \text{of } \mathcal{Q}_{21}\end{array} & \left\{\begin{array}{l}
q_{21}^{11} = C_{2,1}^{1}C_{1,1}^{2} - C_{1,1}^{1}C_{2,1}^{2}\\
q_{21}^{22} = C_{2,2}^{1}C_{1,2}^{2} - C_{1,2}^{1}C_{2,2}^{2}\\
q_{21}^{33} = C_{2,3}^{1}C_{1,3}^{2} - C_{1,3}^{1}C_{2,3}^{2}\\
q_{21}^{44} = C_{2,4}^{1}C_{1,4}^{2} - C_{1,4}^{1}C_{2,4}^{2}
\end{array}\right.\\
\begin{array}{c}\text{Off-diagonal elements}\\ \text{of } \mathcal{Q}_{21}\end{array} & \left\{\begin{array}{l}
q_{21}^{12} = q_{21}^{21} = \dfrac{1}{2}\left(C_{2,1}^{1}C_{1,2}^{2} + C_{2,2}^{1}C_{1,1}^{2} - C_{1,1}^{1}C_{2,2}^{2} - C_{1,2}^{1}C_{2,1}^{2}\right)\\
q_{21}^{13} = q_{21}^{31} = \dfrac{1}{2}\left(C_{2,1}^{1}C_{1,3}^{2} + C_{2,3}^{1}C_{1,1}^{2} - C_{1,1}^{1}C_{2,3}^{2} - C_{1,3}^{1}C_{2,1}^{2}\right)\\
q_{21}^{14} = q_{21}^{41} = \dfrac{1}{2}\left(C_{2,1}^{1}C_{1,4}^{2} + C_{2,4}^{1}C_{1,1}^{2} - C_{1,1}^{1}C_{2,4}^{2} - C_{1,4}^{1}C_{2,1}^{2}\right)\\
q_{21}^{23} = q_{21}^{32} = \dfrac{1}{2}\left(C_{2,2}^{1}C_{1,3}^{2} + C_{2,3}^{1}C_{1,2}^{2} - C_{1,2}^{1}C_{2,3}^{2} - C_{1,3}^{1}C_{2,2}^{2}\right)\\
q_{21}^{24} = q_{21}^{42} = \dfrac{1}{2}\left(C_{2,2}^{1}C_{1,4}^{2} + C_{2,4}^{1}C_{1,2}^{2} - C_{1,2}^{1}C_{2,4}^{2} - C_{1,4}^{1}C_{2,2}^{2}\right)\\
q_{21}^{34} = q_{21}^{43} = \dfrac{1}{2}\left(C_{2,3}^{1}C_{1,4}^{2} + C_{2,4}^{1}C_{1,3}^{2} - C_{1,3}^{1}C_{2,4}^{2} - C_{1,4}^{1}C_{2,3}^{2}\right).
\end{array}\right.
\end{array}\right\} \tag{E.15}
$$

In terms of the coefficients $C_{k,l}^{i}$ given in the expressions (E.3) and (E.4), the elements of $\mathcal{Q}_{23}$ are as follows (see the right-hand side of eq. (E.10)):

$$\begin{array}{ll} \text{Diagonal elements of } \mathcal{Q}_{23} & \left\{ \begin{array}{l} q_{23}^{11} = C_{2,1}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{2,1}^{2} \\ q_{23}^{22} = C_{2,2}^{1}C_{3,2}^{2} - C_{3,2}^{1}C_{2,2}^{2} \\ q_{23}^{33} = C_{2,3}^{1}C_{3,3}^{2} - C_{3,3}^{1}C_{2,3}^{2} \\ q_{23}^{44} = C_{2,4}^{1}C_{3,4}^{2} - C_{3,4}^{1}C_{2,4}^{2} \end{array} \right. \\ \text{Off-diagonal elements of } \mathcal{Q}_{23} & \left\{ \begin{array}{l} q_{23}^{12} = q_{23}^{21} = \frac{1}{2}\left(C_{2,1}^{1}C_{3,2}^{2} + C_{2,2}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{2,2}^{2} - C_{3,2}^{1}C_{2,1}^{2}\right) \\ q_{23}^{13} = q_{23}^{31} = \frac{1}{2}\left(C_{2,1}^{1}C_{3,3}^{2} + C_{2,3}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{2,3}^{2} - C_{3,3}^{1}C_{2,1}^{2}\right) \\ q_{23}^{14} = q_{23}^{41} = \frac{1}{2}\left(C_{2,1}^{1}C_{3,4}^{2} + C_{2,4}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{2,4}^{2} - C_{3,4}^{1}C_{2,1}^{2}\right) \\ q_{23}^{23} = q_{23}^{32} = \frac{1}{2}\left(C_{2,2}^{1}C_{3,3}^{2} + C_{2,3}^{1}C_{3,2}^{2} - C_{3,2}^{1}C_{2,3}^{2} - C_{3,3}^{1}C_{2,2}^{2}\right) \\ q_{23}^{24} = q_{23}^{42} = \frac{1}{2}\left(C_{2,2}^{1}C_{3,4}^{2} + C_{2,4}^{1}C_{3,2}^{2} - C_{3,2}^{1}C_{2,4}^{2} - C_{3,4}^{1}C_{2,2}^{2}\right) \\ q_{23}^{34} = q_{23}^{43} = \frac{1}{2}\left(C_{2,3}^{1}C_{3,4}^{2} + C_{2,4}^{1}C_{3,3}^{2} - C_{3,3}^{1}C_{2,4}^{2} - C_{3,4}^{1}C_{2,3}^{2}\right). \end{array} \right. \end{array} \Bigg\} \qquad \text{(E.16)}$$

In terms of the coefficients $C_{k,l}^{i}$ given in the expressions (E.3) and (E.4), the elements of $\mathcal{Q}_{13}$ are as follows (see the right-hand side of eq. (E.12)):

$$
\left.
\begin{array}{ll}
\begin{array}{c}\text{Diagonal elements}\\ \text{of } \mathcal{Q}_{13}\end{array} &
\left\{
\begin{aligned}
q_{13}^{11} &= C_{1,1}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{1,1}^{2}\\
q_{13}^{22} &= C_{1,2}^{1}C_{3,2}^{2} - C_{3,2}^{1}C_{1,2}^{2}\\
q_{13}^{33} &= C_{1,3}^{1}C_{3,3}^{2} - C_{3,3}^{1}C_{1,3}^{2}\\
q_{13}^{44} &= C_{1,4}^{1}C_{3,4}^{2} - C_{3,4}^{1}C_{1,4}^{2}
\end{aligned}
\right.
\\
\begin{array}{c}\text{Off-diagonal elements}\\ \text{of } \mathcal{Q}_{13}\end{array} &
\left\{
\begin{aligned}
q_{13}^{12} &= q_{13}^{21} = \frac{1}{2}\left(C_{1,1}^{1}C_{3,2}^{2} + C_{1,2}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{1,2}^{2} - C_{3,2}^{1}C_{1,1}^{2}\right)\\
q_{13}^{13} &= q_{13}^{31} = \frac{1}{2}\left(C_{1,1}^{1}C_{3,3}^{2} + C_{1,3}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{1,3}^{2} - C_{3,3}^{1}C_{1,1}^{2}\right)\\
q_{13}^{14} &= q_{13}^{41} = \frac{1}{2}\left(C_{1,1}^{1}C_{3,4}^{2} + C_{1,4}^{1}C_{3,1}^{2} - C_{3,1}^{1}C_{1,4}^{2} - C_{3,4}^{1}C_{1,1}^{2}\right)\\
q_{13}^{23} &= q_{13}^{32} = \frac{1}{2}\left(C_{1,2}^{1}C_{3,3}^{2} + C_{1,3}^{1}C_{3,2}^{2} - C_{3,2}^{1}C_{1,3}^{2} - C_{3,3}^{1}C_{1,2}^{2}\right)\\
q_{13}^{24} &= q_{13}^{42} = \frac{1}{2}\left(C_{1,2}^{1}C_{3,4}^{2} + C_{1,4}^{1}C_{3,2}^{2} - C_{3,2}^{1}C_{1,4}^{2} - C_{3,4}^{1}C_{1,2}^{2}\right)\\
q_{13}^{34} &= q_{13}^{43} = \frac{1}{2}\left(C_{1,3}^{1}C_{3,4}^{2} + C_{1,4}^{1}C_{3,3}^{2} - C_{3,3}^{1}C_{1,4}^{2} - C_{3,4}^{1}C_{1,3}^{2}\right).
\end{aligned}
\right.
\end{array}
\right\}
\qquad \text{(E.17)}
$$

Some simplifications of the elements q_{jk}^{xy} of the coefficient matrices $\mathcal{Q}_{jk}$ in the expressions (E.14)–(E.17) would be possible if we were to substitute the definitions of the coefficients $C_{k,l}^{i}$ from the expressions (E.3) and (E.4). These simplifications are insufficient, however, to justify recasting the elements q_{jk}^{xy} in (E.14)–(E.17) in terms of the underlying primitive parameters. As done above, representing the q_{jk}^{xy} as functions of the coefficients $C_{k,l}^{i}$ is relatively transparent and convenient for our purposes. The MATLAB code used to solve the system in text eqs. (7.25) and (7.26) uses this representation of the problem's parameters in the firms' forward market equilibrium optimality conditions.

E.2 Theory and computation of singularities in the system of text equation (7.32)

In text section 7.2.1, we claimed that the system (7.32) is an example of a *singular quasilinear ODE system*. In this section, we explain why this terminology is appropriate.

Singular ODEs bear a close resemblance to—but are distinct from—so-called *differential-algebraic equations* (DAEs), commonly expressed in the form (Rabier and Rheinboldt 2002, 189)

$$\left.\begin{aligned} \dot{x}_1 &= f(x_1, x_2) \in \mathbb{R}^m \\ 0 &= g(x_1, x_2) \in \mathbb{R}^{n-m} \end{aligned}\right\} \tag{E.18}$$

where $(x_1, x_2) \in \mathbb{R}^m \times \mathbb{R}^{n-m}$.[366] To investigate the distinction between singular ODEs and DAEs, first consider the general form for an implicit ODE,

$$\mathscr{F}(x, \dot{x}) = 0, \tag{E.19}$$

where (letting $r \equiv \dot{x}$ for notational clarity) $\mathscr{F}(x, r): \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is a sufficiently smooth function.[367] If the (partial) derivative $D_r\mathscr{F}(x, r)$ is invertible at a point (x_0, r_0), then eq. (E.19) is clearly reducible to an explicit initial-value problem with initial conditions $x(0) = x_0$ and $\dot{x}(0) = r_0$, and the standard theory of ODEs applies. If on the other hand $D_r\mathscr{F}(x, r)$ is *not* invertible at (x_0, r_0), then eq. (E.19) is either a DAE or a

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[366] As Rabier and Rheinboldt (2002, 190) point out, DAEs need not be clearly divisible into "differential" and "algebraic" components (as is the case in the system (E.18) having *m* differential equations and the $n - m$ algebraic equations). While (E.18) is a familiar form for DAEs from numerous applications, they may also have the more general—implicit—form of $\mathscr{F}(x, \dot{x}) = 0$, as we discuss further below.

[367] This discussion follows closely that of Rabier and Rheinboldt (2002, 190).

singular ODE. The distinction between the two lies in how the singularity of $D_r\mathscr{F}(x,r)$ at (x_0,r_0) affects the total derivative of $\mathscr{F}(x,r)$ with respect to both arguments, denoted as $D\mathscr{F}(x,r)$. Equation (E.19) is classified as a DAE if and only if two conditions hold:

1. The derivative $D\mathscr{F}(x_0,r_0)$ is *surjective*, despite the singularity of $D_r\mathscr{F}(x_0,r_0)$. (Note that when $D_r\mathscr{F}(x_0,r_0)$ is invertible, $D\mathscr{F}(x_0,r_0)$ is surjective, i.e., the function maps onto $\mathbb{R}$).
2. The *rank* of $D_r\mathscr{F}(x_0,r_0)$ is *constant* (and hence not full) on some neighborhood of (x_0,r_0).

In all other cases—particularly when (x_0,r_0) may be approximated arbitrarily closely by points (x,r) at which $D_r\mathscr{F}(x,r)$ is invertible—eq. (E.19) is a *singular ODE*.

Now consider conditions 1 and 2 above with respect to the implicit equation in the text, the system (7.35), for the problem at hand. Whether the first condition above is satisfied depends, in general, on the parameters of the problem, so it will be easier to proceed by examining the second condition above. Replacing x with $\overline{S}^{f++}$ (from eq. (E.5)) and now letting $r \equiv \overline{S}^{f+\prime}(p^f) = \begin{pmatrix}\overline{S}_1^{f\prime}(p^f) & \overline{S}_2^{f\prime}(p^f) & 1\end{pmatrix}^\top$, we may write $\mathscr{F}(x,r)$ as $\mathscr{F}\left(\overline{S}^{f++},r\right)$. Let $p_0^f \in \mathbb{R}$ be a price, and choose the augmented vector of SFs $\overline{S}^{f+}(p^f) \equiv \begin{pmatrix}\overline{S}_1^{f}(p^f) & \overline{S}_2^{f}(p^f) & p^f\end{pmatrix}^\top$ such that the point $\overline{S}^{f+}(p_0^f)$ lies on the singular locus (see text subsection 7.2.1).[368] Next, let $\overline{S}_0^{f++} \equiv \overline{S}^{f++}\left(p_0^f\right)$ be the corresponding doubly-augmented vector of SFs evaluated at p_0^f, and define $r_0 \equiv \overline{S}^{f+\prime}\left(p_0^f\right)$ such that $D_r\mathscr{F}\left(\overline{S}_0^{f++}, r_0\right)$ is singular and of rank k_0. Then, it is clear from the geometry of the singular locus[369] that although $D_r\mathscr{F}\left(\overline{S}_0^{f++}, r_0\right)$ is not invertible, there exist prices $p_{00}^f \neq p_0^f$ arbitrarily close to p_0^f (defining $\overline{S}_{00}^{f++} \equiv \overline{S}^{f++}\left(p_{00}^f\right)$ and $r_{00} \equiv \overline{S}^{f+\prime}\left(p_{00}^f\right)$) for which $D_r\mathscr{F}\left(\overline{S}_{00}^{f++}, r_{00}\right)$ *is* invertible. Hence, there is no neighborhood of $\overline{S}^{f+}\left(p_0^f\right)$ for which the rank of $D_r\mathscr{F}\left(\overline{S}^{f++}\left(p^f\right), r\right)$ is constant at k_0 throughout. We conclude that the second condition above (i.e., that concerning constant rank)—necessary for the system (7.35) in the text to be a DAE—does not hold. This confirms our classification (in text section 7.2.1) of the system (7.35) as a *singular ODE* rather than a DAE.

Singular systems of ODEs are a relatively recent research focus in mathematics.[370] For example, Rabier (1989) was the first systematic study of singular quasilinear (and related) ODE systems (Rabier and Rheinboldt 2002, 324). Rabier and Rheinboldt conjecture (p. 324) that this paucity of attention may be due to a lack of appreciation for the connections between singular DAEs and singular ODEs. Such DAEs do arise naturally, for example, in the theory of electrical networks, in flow problems, and

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368 Here, we have used $\overline{S}_3^f\left(p^f\right) \equiv p^f$ from text eq. (7.29).

369 Recall from text subsection 7.2.1 and Figure 7.1 that the singular locus of the system (7.35) is a *quadratic surface* in $\left(\overline{S}_1^f, \overline{S}_2^f, p^f\right)$-space, a subset of $\mathbb{R}^3$.

370 Rabier and Rheinboldt (2002, 324). These authors also note, however, that the case of *scalar* singular ODEs was analyzed at least as early as 1873.

in plasticity theory.[371] Under a geometric reduction procedure,[372] the DAEs that characterize such phenomena may be recast as singular ODEs, and as such, are often more amenable to analysis. For our purposes, we note that no previous economic applications of singular ODEs are known to the present author, whether in the literature on supply function equilibria or, more broadly, in the fields of game theory or industrial organization. The present investigation thus suggests that the theoretical and numerical tools developed in Rabier and Rheinboldt (2002) may find a new area of application in solving multi-settlement market SFE models.

Presenting the details of Rabier and Rheinboldt's (2002, chs. VII and XIV) analysis of singular quasilinear ODEs would take us too far afield, so we simply state their essential results without proof, insofar as they apply to the system (7.32) in the text, our problem of interest. First, denote a singular point[373]

$$\overline{S}^{f++}\left(p_0^f\right) \equiv \left(\overline{S}_1^f\left(p_0^f\right) \quad \overline{S}_2^f\left(p_0^f\right) \quad p_0^f \quad 1\right)^{\top}$$

at a price p_0^f as a *simple* singular point if the following two conditions hold:[374]

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[371] Among other fields; see Rabier and Rheinboldt (2002, 323) for relevant references.

[372] Rabier and Rheinboldt (2002). See their chapter IV for details of this reduction procedure.

[373] Augmented by "1" as the final element of the vector $\overline{S}^{f++}\left(p_0^f\right)$, for compatibility with system (7.32)–(7.34) in the text.

[374] Regarding these conditions, recall the following definitions from the theory of linear transformations (de la Fuente 2000, 123). Let *X* and *Y* be two vector spaces defined over the same field *F*, and let $T : X \to Y$ be a linear function. Then:

1. The *range* of *T*, $\text{rge}\ T$, is the subset of *Y* given by

$$\text{rge}\ T = T\left(X\right) = \left\{y \in Y : y = T\left(X\right) \text{ for some } x \in X\right\}.$$

2. The *kernel* (or *null space*) of *T*, $\ker T$, is the subset of *X* given by

$$\ker T = T^{-1}\left(\mathbf{0}\right) = \left\{x \in X : T\left(x\right) = \mathbf{0}\right\}.$$

Condition 1: $\dim\ker \mathscr{Q}\left(\overline{S}^{f++}\left(p_0^f\right)\right)=1$

Condition 2: $\mathscr{G}\left(\overline{S}^{f++}\left(p_0^f\right)\right)\notin \operatorname{rge}\mathscr{Q}\left(\overline{S}^{f++}\left(p_0^f\right)\right)$

Consider whether text eqs. (7.32)–(7.34) that characterize our problem satisfy these two conditions. We argue, first, that *no* point in the singular locus of the system (7.32) in the text satisfies Condition 1 above. This is because at every singular point in our problem, both text eqs. (7.36) and (7.37) hold, implying that

$$\dim\ker \mathscr{Q}\left(\overline{S}^{f++}\left(p_0^f\right)\right)=2\,. \tag{E.20}$$

Therefore, even before considering Condition 2, we may conclude—since Condition 1 is violated everywhere—that points in the singular locus of the system (7.32) in the text are *not* simple singular points. Examining Condition 2 above for completeness' sake, this condition implies that any point lying on the manifold at which the singular locus intersects the graph of either of the first two terms of the vector $\mathscr{G}\left(\overline{S}^{f++}\left(p_0^f\right)\right)$[375] is also not a simple singular point. That is, Condition 2 requires that, for $\overline{S}^{f++}\left(p_0^f\right)$ to be a simple singular point, it is necessary that

$$\overline{S}^{f++}\left(p_0^f\right)^{\top}\mathscr{Q}_{23}\overline{S}^{f++}\left(p_0^f\right)\neq 0 \tag{E.21}$$

and

In words, $\operatorname{rge} T$ is the set of vectors $y\in Y$ for which $T\left(X\right)=y$ has at least one solution, while $\ker T$ is the set of solutions to the homogeneous linear system $T\left(x\right)=\mathbf{0}$.

[375] Recall that these graphs are also quadratic forms.

$$\overline{S}^{f++}\left(p_0^f\right)^{\top}\mathcal{Q}_{13}\overline{S}^{f++}\left(p_0^f\right)\neq 0\,. \tag{E.22}$$

Most, but not all, singular points in the present problem do satisfy eqs. (E.21) and (E.22), as text Table 7.1 explains.[376]

Rabier and Rheinboldt's (2002, 330–31) Theorem 39.1 is an existence theorem for solutions to singular ODEs in the neighborhood of simple singular points. It posits the existence of two distinct solutions in such a neighborhood whose (joint) graph does not cross (i.e., is not transverse to) the singular locus. In contrast, the existence theory for solutions to singular ODEs in the neighborhood of singular points that are not simple, according to Rabier and Rheinboldt (2002, 331), "is much more involved and virtually untouched in the published literature. . . . The problems when [$\dim\ker\mathcal{A}\left(\overline{S}^{f++}\left(p_0^f\right)\right)\geq 2$] or [$\mathcal{G}\left(\overline{S}^{f++}\left(p_0^f\right)\right)\in \operatorname{rge}\mathcal{A}\left(\overline{S}^{f++}\left(p_0^f\right)\right)$] are open." Recalling eq. (E.20), this statement applies to the system (7.32) in the text.

Apart from existence theory for solutions, Rabier and Rheinboldt (2002, ch. XIV) also outline a computational approach for solving singular ODEs. Their procedure is based on computational methods for nonlinear algebraic equations, that is, equation systems lacking a dynamic component. This procedure exploits a reparameterization of the problem that renders the equations computable in the neighborhood of the (erstwhile) singularity. It was originally developed in earlier work by these authors (Rabier and Rheinboldt 1994a, 1994b), and remarkably, is applicable not only to simple singular points, but also to more complex singularities (Rabier and Rheinboldt 2002, 483). The

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376 Condition 2 above is, of course, superfluous in this case since points on our singular locus do not satisfy Condition 1. As noted above, the singular locus contains only singular points that are not simple.

algorithm has not yet, to the author's knowledge, been implemented using standard numerical analysis software packages such as MATLAB, Maple, or Mathematica.[377] Such an effort would be worthwhile to the extent that computing solutions near the singularities of systems such as (7.32) in the text is of concern.

E.3 The MATLAB ode15s solver

The MATLAB ODE solver used in this investigation is named ode15s. This solver performed quite well for the qualitative and numerical investigations of text chapter 7, permitting the author to compute SF trajectories quite close to the singular locus. Nonetheless, it is important to keep in mind when interpreting the MATLAB-based results of text chapter 7 that Rabier and Rheinboldt's procedure for solving singular ODEs discussed in section E.2 above is not reflected in the solver ode15s. That is, the algorithm in ode15s may not be fully robust in the presence of singularities. As a consequence, in the neighborhood of singularities, it is a priori unclear whether a particular trajectory reflects underlying theoretical properties of the singular ODE system, or whether characteristics of an SF trajectory might only be artifacts of the solver algorithm itself. Because this investigation does not explore in detail trajectories' behavior in the neighborhood of singularities, we need not explore this issue further here.

The ode15s solver performed best with the "backwards differentiation" option enabled, which exploits the so-called backwards differentiation formulae (BDFs). To

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[377] The author is indebted to Werner Rheinboldt for making available some FORTRAN codes—still under development—for solution of singular DAEs. The performance of these codes on the problem at hand has not yet been investigated; this, too, is a matter for future research.

understand the essentials of the BDFs, first define $\overline{S}^f = \overline{S}^f\left(p^f\right)$ as the vector of firms' supply functions:

$$\overline{S}^f = \overline{S}^f\left(p^f\right) \equiv \left(\overline{S}_1^f\left(p^f\right) \quad \overline{S}_2^f\left(p^f\right)\right)^{\top}. \tag{E.23}$$

Next, express the system (7.40)–(7.42) in the text in vector form as

$$\overline{S}^{f\prime}\left(p^f\right) = g\left(\overline{S}^f, p^f\right).$$

Index the iterates in the numerical approximation to the trajectory $\overline{S}^f\left(p^f\right)$ with a superscript "^{t}," and thus write the t^{th} iterate of this approximation as

$$\left(\overline{S}^{f,t}, p^{f,t}\right).$$

Next, define the *backward difference operator* of *order* $j \geq 0$, ∇^j, inductively as follows:

$$\nabla^0 \overline{S}^{f,t} = \overline{S}^{f,t};$$

$$\nabla^{j+1} \overline{S}^{f,t} = \nabla^j \overline{S}^{f,t} - \nabla^j \overline{S}^{f,t-1}.$$

For a particular $t = \hat{t}$, the implicit formula for $\left(\overline{S}^{f,\hat{t}+1}, p^{f,\hat{t}+1}\right)$ given the $\hat{t}+1$ iterates $\left\{\left(\overline{S}^{f,t}, p^{f,t}\right)_{t=0}^{t=\hat{t}}\right\}$[378] and a step size h is (Shampine and Reichelt 1997, 2)

$$\sum_{m=1}^{k} \frac{1}{m} \nabla^m \overline{S}^{f,\hat{t}+1} - h \cdot g\left(\overline{S}^{f,\hat{t}+1}, p^{f,\hat{t}+1}\right) = 0. \tag{E.24}$$

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[378] The first such iterate, of course, is a given initial condition $\left(\overline{S}^{f,0}, p^{f,0}\right)$.

The MATLAB ode15s solver approximates the implicit nonlinear equation (E.24) with simplified Newton iteration starting with the predicted value

$$\overline{S}^{f,\hat{\imath}+1} = \sum_{m=0}^{k} \nabla^{m} \overline{S}^{f,\hat{\imath}} .$$

The routine ode15s is a *variable order* solver, meaning that the solver varies the order k of the finite differences used to compute $\overline{S}^{f,\hat{\imath}+1}$ via eq. (E.24).[379] The choice of order entails, in general, a tradeoff between efficiency (speed of computation) and stability (roughly speaking, the property that small perturbations in the initial condition lead to small deviations in the sequence of iterates). In this work, we find via experimentation that a maximum order of $k_{\max} = 5$ consistently produces stable solutions of the system (7.40)–(7.42) in the text, so we use this value as the default maximum order for all results.

Shampine and Reichelt (1997, 2) characterize the routine ode15s as having a "quasi-constant" step size h (see eq. (E.24)). By this they mean that "the step size is held constant [by the solver] during an integration unless there is good reason to change it." The good reason, in this instance, would be to maintain the local discretization error within desired tolerances (see note 380 below). In text subsection 7.4.2, for example, we noted that when the SF trajectory is "absorbed" by the ∞-locus (as with trajectory ③ in text Figure 7.5), the MATLAB solver fails and numerical integration halts. Here, solver failure indicates that it is no longer numerically feasible for the solver to maintain simultaneously the following two conditions:

[379] In addition, MATLAB permits the user to specify the maximum order $k_{\max}$ to be equal to any integer between 1 and 5 (inclusive).

1. enforce the chosen tolerance on numerical errors[380] by selecting a sufficiently small step size, and
2. keep the step size large enough[381] to make "acceptable" progress in the integration.

E.4 Numerical results of comparative statics analysis

Table 7.2 in text subsection 7.6.3 reports the qualitative effects on firms' forward market quantities, $\overline{q}_1^f = \overline{S}_1^f\left(p^f\right)$ and $\overline{q}_2^f = \overline{S}_2^f\left(p^f\right)$, of perturbing each of the ten elements in the parameter vector Θ; we refer to the study of such effects as *comparative statics analysis* (see text section 7.6). The qualitative effects documented in text Table 7.2 are based on the numerical results of chapter 7's discrete Excel model. This section reports these numerical results.

Table E.1 below summarizes the comparative statics results produced by the discrete Excel model. Each test case reported in the table has row headings—for the top

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[380] Any MATLAB solver (again, we use the solver ode15s) will compute a numerical approximation to the true solution $\left(\overline{S}_1^f\left(p^f\right), \overline{S}_2^f\left(p^f\right), p^f\right)$ to the system in text eqs. (7.40)–(7.42). Naturally, numerical error is inherent in this numerical approximation. Numerical errors are of two types, *discretization error*, and *roundoff error* (Moler and Moler 2003, section 6.13). The former depends on the underlying differential equation system and the chosen numerical method, while the latter is a function of computer software and hardware. Using current computing platforms, roundoff error is only likely to become important if very high accuracies are requested or the interval of integration is very large. Through adjustments of the step size *h*, the solver algorithm controls the (local) discretization error (related to the order *k* of the numerical method), maintaining it within prescribed tolerances. The higher the order, the smaller the local discretization error. As discussed above, we use the highest possible order, $k = 5$, in the BDFs. Given a step size h_t for step t, the local discretization error is $O\left(h_t^{k+1}\right) = O\left(h_t^6\right)$, which is likely to be acceptably small. We used MATLAB's default error tolerances for relative and absolute error; see the program's documentation (The MathWorks 2001) for details. For in-depth treatments of error analysis in numerical integration, consult Butcher (1987) or Hairer, Noersett and Wanner (1993).

[381] The minimum step size is a parameter in MATLAB's ODE solvers that the model user may vary. The default minimum step size is $\sim 10^{-14}$.

four rows in the table—labeled "θ," "θ^{base}," "δ^{mult}," and "θ^{test}" (see the upper left corner of Table E.1). We define these headings as follows:

- θ: Comparative statics parameter (that is, the elements of Θ: c_{01}, c_{02}, and so forth; see Table 7.2 in the text for an explanation of each parameter)
- θ^{base}: The base case value of each comparative statics parameter θ, based on empirical data from the California PX, *circa* 1999 (see Appendix F for details).
- δ^{mult}: The multiplicative shock relating the base case parameter value to the test case parameter value (see eq. (E.26) below). The results of Table E.1 below assume that

$$\delta^{mult} = 1.001 \tag{E.25}$$

for each comparative statics scenario, corresponding to a 0.1% increase in the parameter under consideration.[382]

- θ^{test}: The test case value of the comparative statics parameter θ, that is,

$$\theta^{test} = \delta^{mult} \cdot \theta^{base}. \tag{E.26}$$

As explained below, the body of Table E.1 has a pair of columns corresponding to each of the ten test cases in which we perturb the parameters in Θ (c_{01}, c_{02}, and so forth)

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[382] Recall that each parameter in Θ enters the underlying equilibrium optimality conditions (text eqs. (7.11) and (7.12)) highly nonlinearly and through multiple pathways. It is therefore not surprising that not only the magnitude but also the sign of the comparative statics effects on $\overline{S}_i^f\left(p^f\right)$ can vary with sufficiently large variations in the magnitude of the multiplicative shock δ^{mult}. The sign of the comparative statics effects reported in Table E.1 below are valid at least for small to moderate shocks in the interval $\delta^{mult} \in [1.001, 1.01]$ (i.e., shocks of 0.1% to 1% of base case values), and usually for a much larger range of δ^{mult}.

individually, as well as a column of base case values (the third column of the table) for β_i^s and the discretized SFs $\bar{S}_i^f\left(p^f\right)$. The rows in the body of Table E.1 comprise three sections, as follows:

- The upper section of the body of the table (consisting of two rows only) contains values of the spot market SF slopes β_1^s and β_2^s for the base case and each test case, including (in the columns labeled "Δ") the absolute change in β_i^s between the test case and the base case.
- The middle and lower sections of the body of the table contain, respectively, quantities (in MWh) defining firm 1's and firm 2's discretized SFs in the price range of $p^f \in [0,\ 2{,}750]\ \$/\mathrm{MWh}$ with a step size of $\Delta p^f = \$250/\mathrm{MWh}$. For ease of readability, we denote firm *i*'s discretized SF as "*Si_t*" in the table. Here, $i = 1, 2$ indexes the supplier firms, while $t = 0, 1, 2, \ldots, 11$ indexes the points, or quantities, at which we evaluate each firm's discrete SF. (The notation *Si*_0 represents, naturally, firm *i*'s initial quantity).

The middle and lower sections of the columns labeled "Δ" in Table E.1 give—for firms 1 and 2, respectively—the absolute change in the respective discretized SF (i.e., change in quantity) between the test case and the base case. For example, in the section of Table E.1 corresponding to c_{02} (on the first page of the table), consider the point on the discretized forward market SF "*S1_2*" at $p^f = \$500/\mathrm{MWh}$, or $\bar{S}_1^f(500)$, in our customary notation. For this point on firm 1's SF, we have that (to four decimal places)

$$\begin{aligned}\Delta &= \bar{S}_1^f\left(500\right)\Big|_{\text{Test case}} - \bar{S}_1^f\left(500\right)\Big|_{\text{Base case}}\\ &= 2723.0530\text{ MWh} - 2722.9917\text{ MWh}\\ &= 0.0613\text{ MWh.}\end{aligned}$$

For purposes of the comparative statics analysis, we ignore in Table E.1 below the lowest and highest points on each discretized SF, since the first- and second-order optimality conditions for the SFs are not imposed at these points.[383] Finally, the abbreviation "NS" used in the row of headings in Table E.1 denotes "No Scaling." This designation indicates that automatic scaling was not used in the discrete Excel model in producing a comparative statics scenario that is so labeled (see note 284 in chapter 7).[384]

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[383] That is, we ignore the points *Si_0* and *Si_11* ($i = 1, 2$) on the discretized SFs at prices $p^f = \$0/\text{MWh}$ and $p^f = \$2{,}750/\text{MWh}$, respectively.

[384] As it happens, we did not use scaling in any of the scenarios reported in Table E.1.

TABLE E.1: COMPARATIVE STATICS RESULTS

θ	**Spot mkt.**		c_{01}	Δ	c_{02}	Δ	c_1	Δ	c_2	Δ	e^s_{dem}	Δ
θ^{base}	**slopes &**	**Base case**	25.6		30.5		0.000341		0.00326		-5.95E-05	
δ^{mult}	**fwd. mkt.**	**values**	1.001	NS	1.001	NS	1.001	NS	1.001	NS	1.001	NS
θ^{test}	**quantities**		25.6256		30.5305		0.000341		0.003263		-5.96E-05	
	β^s_1	2.4830	2.4830	0.0000	2.4830	0.0000	2.4829	-0.0001	2.4819	-0.0011	2.4843	0.0012
p^f	β^s_2	2.4740	2.4740	0.0000	2.4740	0.0000	2.4739	-0.0001	2.4729	-0.0011	2.4752	0.0012
0	*S1_0*	2551.4527	2551.4454	-0.0073	2551.4972	0.0445	2551.3602	-0.0925	2550.6716	-0.7812	2628.2261	76.7734
250	*S1_1*	2551.4527	2551.4454	-0.0073	2551.4972	0.0445	2551.3602	-0.0925	2550.6716	-0.7812	2628.2261	76.7734
500	*S1_2*	2722.9917	2723.0068	0.0151	2723.0530	0.0613	2722.9154	-0.0764	2722.3093	-0.6825	2809.5458	86.5540
750	*S1_3*	2865.6312	2865.6551	0.0239	2865.6961	0.0649	2865.5474	-0.0838	2864.9150	-0.7162	2958.9225	93.2913
1000	*S1_4*	2969.8024	2969.8395	0.0370	2969.8767	0.0743	2969.7328	-0.0697	2969.1800	-0.6224	3070.2596	100.4572
1250	*S1_5*	3050.0288	3050.0698	0.0409	3050.1042	0.0754	3049.9548	-0.0740	3049.4057	-0.6231	3156.2485	106.2197
1500	*S1_6*	3103.8514	3103.9018	0.0504	3103.9337	0.0823	3103.7917	-0.0597	3103.3242	-0.5272	3215.1220	111.2706
1750	*S1_7*	3145.8162	3145.8685	0.0523	3145.8989	0.0827	3145.7542	-0.0620	3145.3101	-0.5061	3262.4098	116.5936
2000	*S1_8*	3167.7511	3167.8112	0.0600	3167.8395	0.0884	3167.7037	-0.0475	3167.3407	-0.4105	3287.8595	120.1083
2250	*S1_9*	3184.9707	3185.0314	0.0608	3185.0591	0.0885	3184.9223	-0.0484	3184.5973	-0.3734	3310.2982	125.3275
2500	*S1_10*	3184.9707	3185.0382	0.0676	3185.0640	0.0933	3184.9369	-0.0338	3184.6912	-0.2795	3312.5914	127.6207
2750	*S1_11*	3184.9707	3185.0382	0.0676	3185.0640	0.0933	3184.9369	-0.0338	3184.7409	-0.2298	3317.8970	132.9264
0	*S2_0*	629.5157	629.5017	-0.0140	629.4905	-0.0253	629.5360	0.0203	629.5952	0.0794	645.7125	16.1968
250	*S2_1*	1863.9420	1863.9943	0.0524	1863.9563	0.0144	1863.8914	-0.0506	1863.4615	-0.4805	1918.8229	54.8809
500	*S2_2*	2326.6708	2326.7374	0.0665	2326.7062	0.0353	2326.6190	-0.0519	2326.1632	-0.5077	2401.0999	74.4290
750	*S2_3*	2588.6977	2588.7650	0.0673	2588.7393	0.0416	2588.6312	-0.0666	2588.0913	-0.6064	2672.3705	83.6727
1000	*S2_4*	2760.6702	2760.7434	0.0733	2760.7240	0.0539	2760.6135	-0.0566	2760.1432	-0.5270	2855.3364	94.6662
1250	*S2_5*	2878.1138	2878.1868	0.0730	2878.1705	0.0567	2878.0506	-0.0633	2877.5487	-0.5651	2977.4691	99.3553
1500	*S2_6*	2963.5659	2963.6436	0.0777	2963.6320	0.0661	2963.5150	-0.0509	2963.1085	-0.4574	3071.7044	108.1384
1750	*S2_7*	3020.6595	3020.7368	0.0774	3020.7270	0.0675	3020.6053	-0.0542	3020.1860	-0.4734	3131.3366	110.6771
2000	*S2_8*	3063.2945	3063.3759	0.0814	3063.3699	0.0754	3063.2535	-0.0410	3062.9431	-0.3514	3181.9061	118.6116
2250	*S2_9*	3086.0252	3086.1064	0.0812	3086.1011	0.0759	3085.9829	-0.0423	3085.6684	-0.3568	3205.7474	119.7222
2500	*S2_10*	3102.6736	3102.7582	0.0847	3102.7565	0.0829	3102.6448	-0.0288	3102.4485	-0.2251	3230.0090	127.3355
2750	*S2_11*	3102.6736	3102.7582	0.0847	3102.7565	0.0829	3102.6448	-0.0288	3102.4485	-0.2251	3230.0090	127.3355

Notes:

NS: No automatic scaling used to produce scenario.

TABLE E.1: COMPARATIVE STATICS RESULTS (CONT'D)

$\overline{\eta}_R$	Δ	$\sigma^2_{\eta_R}$	Δ	$\overline{\nu}_R$	Δ	$\sigma^2_{\nu_R}$	Δ	λ_R	Δ	
4642.3791		2456747.9		334.59885		58604.631		0.0003198		
1.001	NS	1.001	NS	1.001	NS	1.001	NS	1.001	NS	
4647.0214		2459204.7		334.93345		58663.235		0.0003202		
2.4830	0.0000	2.4830	0.0000	2.4830	0.0000	2.4830	0.0000	2.4830	0.0000	
2.4740	0.0000	2.4740	0.0000	2.4740	0.0000	2.4740	0.0000	2.4740	0.0000	**Price vector**
2552.8893	1.4366	2627.4641	76.0114	2551.9295	0.4768	2549.7048	-1.7479	2550.5097	-0.9430	0
2552.8893	1.4366	2627.4641	76.0114	2551.9295	0.4768	2549.7048	-1.7479	2550.5097	-0.9430	250
2724.7653	1.7735	2808.9555	85.9637	2723.7001	0.7083	2721.1081	-1.8837	2722.1953	-0.7964	500
2867.4573	1.8261	2958.2769	92.6457	2866.4135	0.7823	2863.6329	-1.9983	2864.7765	-0.8548	750
2971.8791	2.0766	3069.7653	99.9629	2970.7493	0.9469	2967.7328	-2.0696	2969.0773	-0.7251	1000
3052.1331	2.1043	3155.7324	105.7036	3051.0146	0.9858	3047.9022	-2.1266	3049.2758	-0.7530	1250
3106.1602	2.3088	3214.7552	110.9038	3104.9680	1.1166	3101.6956	-2.1558	3103.2282	-0.6232	1500
3148.1387	2.3226	3262.0430	116.2268	3146.9528	1.1367	3143.6365	-2.1797	3145.1848	-0.6313	1750
3170.2502	2.4990	3287.6404	119.8892	3169.0005	1.2493	3165.5672	-2.1839	3167.2498	-0.5014	2000
3187.4751	2.5045	3310.0944	125.1237	3186.2283	1.2576	3182.7837	-2.1870	3184.4747	-0.4959	2250
3187.6325	2.6618	3312.5332	127.5626	3186.3301	1.3594	3182.7959	-2.1747	3184.6043	-0.3664	2500
3187.6325	2.6618	3317.8659	132.8952	3186.3301	1.3594	3182.8071	-2.1636	3184.6199	-0.3508	2750
630.1164	0.6007	645.9504	16.4347	627.1837	-2.3320	630.5148	0.9990	629.3678	-0.1480	0
1865.0770	1.1351	1918.4134	54.4714	1863.4250	-0.5170	1863.1679	-0.7740	1863.2790	-0.6630	250
2328.2668	1.5960	2400.7128	74.0420	2326.8055	0.1346	2325.3474	-1.3234	2326.0355	-0.6354	500
2590.3978	1.7000	2671.8600	83.1623	2589.0877	0.3899	2587.0875	-1.6102	2587.9514	-0.7463	750
2762.6497	1.9795	2854.9505	94.2803	2761.3111	0.6410	2758.8963	-1.7739	2760.0304	-0.6398	1000
2880.1372	2.0234	2977.0308	98.9170	2878.8619	0.7480	2876.2256	-1.8882	2877.4243	-0.6895	1250
2965.8076	2.2417	3071.4129	107.8469	2964.4720	0.9061	2961.6097	-1.9563	2963.0012	-0.5648	1500
3022.9216	2.2621	3131.0207	110.3613	3021.6285	0.9690	3018.6511	-2.0084	3020.0707	-0.5887	1750
3065.7422	2.4477	3181.7463	118.4518	3064.3809	1.0864	3061.2607	-2.0337	3062.8382	-0.4563	2000
3088.4808	2.4556	3205.5783	119.5531	3087.1556	1.1304	3083.9715	-2.0536	3085.5595	-0.4657	2250
3105.2940	2.6204	3230.0014	127.3278	3103.8980	1.2244	3100.6189	-2.0547	3102.3446	-0.3289	2500
3105.2940	2.6204	3230.0014	127.3278	3103.9325	1.2590	3100.6189	-2.0547	3102.3446	-0.3289	2750

> [T]he trouble about arguments is, they ain't nothing but THEORIES, after all, and theories don't prove nothing, they only give you a place to rest on, a spell, when you are tuckered out butting around and around trying to find out something there ain't no way TO find out. . . . There's another trouble about theories: there's always a hole in them somewheres, sure, if you look close enough.
>
> —Mark Twain, *Tom Sawyer Abroad*

Appendix F: Base case parameter values used in the numerical examples of the multi-settlement SFE model

THIS APPENDIX explains the provenance of the base case parameter values used for the qualitative and quantitative analysis of text chapter 7. As the citations below suggest, the chosen parameter values are based (very roughly) on California's fossil-fired generation capacity during the interval June 1998 to September 1999, which we call the "reference period."[385] Generating units in the California market which must run due to engineering constraints were bid into the PX with a (perfectly elastic and non-strategic) bid of zero dollars; these units were largely those using non-fossil fuel generation technologies: hydroelectric, nuclear, and geothermal plants. These units almost never set the market-

[385] Unless otherwise noted, data reported apply to this reference period.

clearing price and might as a first approximation (following Borenstein, Bushnell and Wolak 2002) be treated as bidding non-strategically. The analysis that follows nets out the load served by these non-fossil fuel units and focuses on the fraction of the market served by fossil-fired units.

We discuss spot market parameters in section F.1 below, followed by those parameters relevant to the forward market in section F.2. In closing, section F.3 summarizes the numerical findings of this appendix in the base case parameter vector Θ^{base}.

F.1 Spot market

Figure F.1 below depicts firms' marginal cost functions $C_i'\left(q_i^s\right)$ and the spot market demand function $D^s\left(p^s, \varepsilon^s\right)$ given a demand shock ε^s, using text chapter 5's affine assumptions. The figure also depicts an empirical reference price $p_{empir}^{s,mean}$ and empirical (aggregate) reference quantity $q_{empir}^{s,mean}$ for the spot market. Subsection F.1.1 below provides values of $p_{empir}^{s,mean}$ and $q_{empir}^{s,mean}$ from the literature.

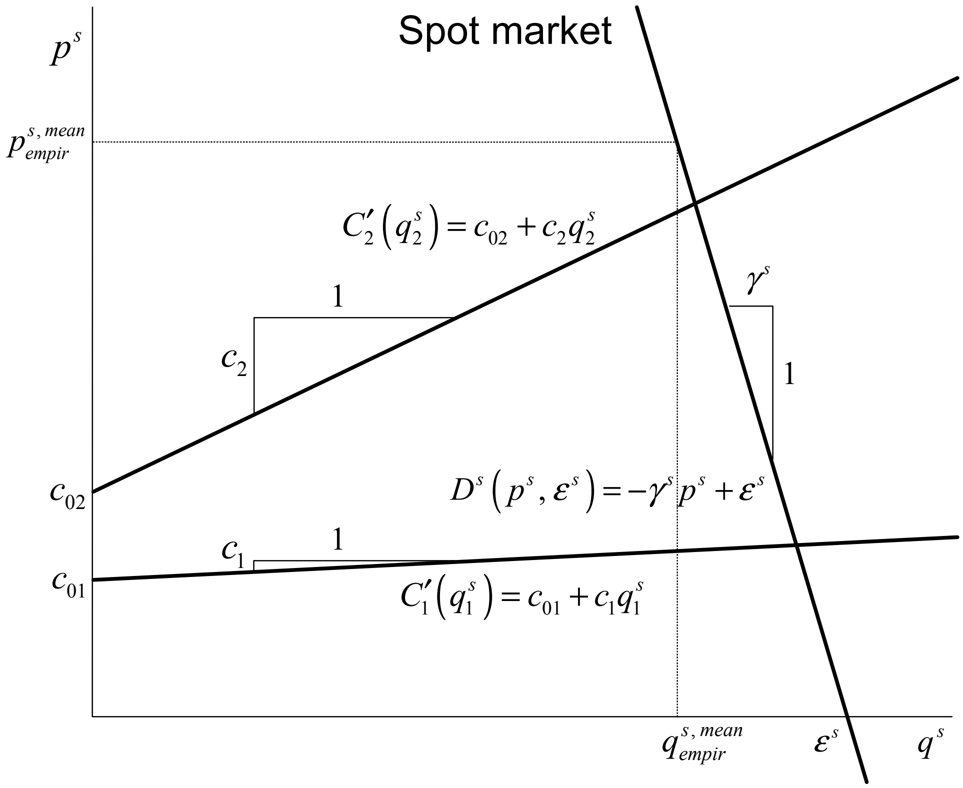

FIGURE F.1: SPOT MARKET GEOMETRY

F.1.1 Prices and quantities

The following parameters are available directly from the literature:

- $p_{empir}^{s,\,mean} = \$26.54/\text{MWh}$: Mean California ISO spot market price during the reference period, averaged over all hours and the two zones NP15 and SP15 (Borenstein, Bushnell and Wolak 2000, 13)
- $q_{empir}^{s,\,mean} = 4{,}955\ \text{MWh}$: Mean aggregate spot market demand facing fossil-fired units in the California ISO system during the reference period, averaged over all hours (Bushnell 2003a)

- $q^{s,mean}_{empir,tot} = 26{,}511\,\text{MWh}$: Mean aggregate spot market demand facing *all* generating units in the California ISO system during the reference period, averaged over all hours (Borenstein, Bushnell and Wolak 2002, 1393)

F.1.2 Demand data

In text chapter 7, we denoted as e^{s}_{dem} the price elasticity of spot market demand facing fossil-fired units in the California ISO system during the reference period (evaluated at the empirical reference price $p^{s,mean}_{empir}$ and empirical (aggregate) reference quantity $q^{s,mean}_{empir}$). One empirically-based approximation of the spot market demand elasticity is Bushnell and Mansur's (2002, 19) estimate of $e^{s}_{dem} = -0.02$.[386] Unfortunately, this value of e^{s}_{dem} did not lead to feasible solutions of the discrete Excel model when applied to the benchmarking procedure of text section 7.5. As a consequence, we permitted e^{s}_{dem} to be *endogenous* in the benchmarking procedure, and describe here how we obtained the value of e^{s}_{dem} ultimately used in the analysis of text chapter 7.

The benchmarking procedure of text section 7.5 centers around a sequence of two optimization problems:

1. Benchmarking step 1 (text problem (7.55))
2. Benchmarking step 2 (text problem (7.56))

---

[386] Bushnell and Mansur caution against interpreting the results of their calculations as elasticities, per se, since retail prices to consumers were being deregulated during the period that they study, and the price that consumers *thought* that they faced as consumption decisions were made is, of course, unobserved. Moreover, these authors' empirical work is based on data only for the San Diego area during the period August and September 2000, rather than data for the California market as a whole.

Trial and error with variants of text problem (7.55) leads to using $e_{dem}^{s} = -0.0015411$ as an element of the reduced parameter vector $\Theta^{(0)} \setminus \left(\overline{\eta}_R, \sigma_{\eta_R}^2, \overline{\nu}_R, \sigma_{\nu_R}^2\right)$ in this problem, and hence also as the initial value of e_{dem}^{s} in text problem (7.56). The solution to text problem (7.56) in step 2 of the benchmarking procedure yields an endogenous value of e_{dem}^{s},

$$\left(e_{dem}^{s}\right)^{(2)} = -5.9507\text{e-}5, \tag{F.1}$$

that we may then incorporate into the base case parameter vector Θ^{base}. The elasticity in eq. (F.1) is practically equal to zero, and hence is probably smaller in magnitude than would be realistic for the California electricity market. It is, however, the endogenous value of e_{dem}^{s} that yielded the best fit of prices and quantities in the benchmarking procedure.

Assuming affine spot market demand as in the simplified affine example of text chapter 5, and using values of $\left(e_{dem}^{s}\right)^{(2)}$, $p_{empir}^{s,\,mean}$, and $q_{empir}^{s,\,mean}$ from this and the previous subsection, we may compute the corresponding slope γ^{s} of the affine spot market demand function facing fossil-fired units as[387] (to five significant figures)

$$\gamma^{s} = -\frac{dq^{s}}{dp^{s}} = -\frac{q_{empir}^{s,\,mean}\left(e_{dem}^{s}\right)^{(2)}}{p_{empir}^{s,\,mean}} = -\frac{\left(4{,}955\ \text{MWh}\right)\left(-5.9507\text{e-}5\right)}{\left(26.54\ \dfrac{\$}{\text{MWh}}\right)} = 0.011110\,\frac{\text{MWh}}{\$/\text{MWh}}. \tag{F.2}$$

[387] Minor discrepancies in numerical results are due to rounding.

F.1.3 Cost data

We model the aggregate marginal cost function for fossil-fired units in the California ISO system during the reference period as comprising only *two* (hypothetical) firms, labeled 1 and 2, in accordance with the duopoly model developed in the thesis. Based on the aggregate marginal cost function for fossil-fired units in Figure 1 of Borenstein, Bushnell, and Wolak (2000), we find—graphically—the following parameter values for the intercepts and slopes of the two hypothetical firms' marginal cost functions:

$$c_{01} = 25.6 \frac{\$}{\text{MWh}}; \tag{F.3}$$

$$c_{02} = 30.5 \frac{\$}{\text{MWh}}; \tag{F.4}$$

$$c_1 = 0.000341 \frac{\$/\text{MWh}}{\text{MWh}}; \tag{F.5}$$

and

$$c_2 = 0.00326 \frac{\$/\text{MWh}}{\text{MWh}}. \tag{F.6}$$

The parameter values in eqs. (F.3)–(F.6) imply that firm 1 is a "low-cost" firm and firm 2 a "high-cost" firm in the sense that $c_1 < c_2$ and $c_{01} < c_{02}$. We use these values from eqs. (F.3)–(F.6) in the base case parameter vector Θ^{base}.

F.1.4 Spot market SF slopes and related parameters

Recalling the analysis of text section 5.2, we may solve the pair of equations

$$\beta_i^s = \frac{\left(\gamma^s + \beta_j^s\right)}{1 + c_i\left(\gamma^s + \beta_j^s\right)} \quad \left(i, j = 1, 2;\ i \neq j\right)$$

for the slopes β_1^s and β_2^s of the spot market SFs. Evaluating these slopes at the values γ^s, c_1, and c_2 from eqs. (F.2), (F.5), and (F.6) above, we have (to five significant figures)

$$\beta_1^s = 2.4830 \frac{\text{MWh}}{\$/\text{MWh}} \tag{F.7}$$

and

$$\beta_2^s = 2.4740 \frac{\text{MWh}}{\$/\text{MWh}}. \tag{F.8}$$

Given eqs. (F.2) and (F.3)–(F.8), we may compute ω_a and ω_b (see text eqs. (5.24) and (5.25)) as

$$\begin{aligned}\omega_a &= \frac{1}{\beta_1^s + \beta_2^s + \gamma^s} \\ &= \frac{1}{2.4830 \frac{\text{MWh}}{\$/\text{MWh}} + 2.4740 \frac{\text{MWh}}{\$/\text{MWh}} + 0.011110 \frac{\text{MWh}}{\$/\text{MWh}}},\end{aligned}$$

or

$$\omega_a = 0.20128 \frac{\$/\text{MWh}}{\text{MWh}}, \tag{F.9}$$

and

$$\begin{aligned}\omega_b &= c_{01}\beta_1^s + c_{02}\beta_2^s \\ &= \left(25.6 \frac{\$}{\text{MWh}}\right)\left(2.4830 \frac{\text{MWh}}{\$/\text{MWh}}\right) + \left(30.5 \frac{\$}{\text{MWh}}\right)\left(2.4740 \frac{\text{MWh}}{\$/\text{MWh}}\right),\end{aligned}$$

or

$$\omega_b = 139.02 \text{ MWh} . \tag{F.10}$$

F.1.5 Distributional assumptions for spot market demand

The solution to text problem (7.56) corresponding to step 2 of the benchmarking procedure yields optimal values of the mean $\bar{\eta}_R$ and variance $\sigma^2_{\eta_R}$ of the representative consumer's signal η_R, as well as the mean $\bar{\nu}_R$ and variance $\sigma^2_{\nu_R}$ of the spot market noise parameter ν_R. To five significant figures, these optimal values are as follows:

$$\left(\bar{\eta}_R\right)^{(2)} = 4{,}642.4 \text{ MWh} , \tag{F.11}$$

$$\left(\sigma^2_{\eta_R}\right)^{(2)} = 2.4567\text{e}6 \text{ MWh}^2 , \tag{F.12}$$

$$\left(\bar{\nu}_R\right)^{(2)} = 334.60 \text{ MWh} , \tag{F.13}$$

and

$$\left(\sigma^2_{\nu_R}\right)^{(2)} = 58{,}605 \text{ MWh}^2 . \tag{F.14}$$

From eqs. (F.12) and (F.14), the standard deviations of η_R and ν_R are

$$\left(\sigma_{\eta_R}\right)^{(2)} = 1{,}567.4 \text{ MWh} \tag{F.15}$$

and

$$\left(\sigma_{\nu_R}\right)^{(2)} = 242.08 \text{ MWh} . \tag{F.16}$$

The values in eqs. (F.11)–(F.14) are optimal in the sense that they solve text problem (7.55), ensuring also that the two benchmarking constraints $\mathrm{E}\left(p^s\right) = p^{s,\,mean}_{empir}$ and

$\mathrm{E}\left(\overline{q}_{Agg}^{s}\right)=q_{empir}^{s,mean}$ imposed in that problem are met. Finally, we incorporate the optimal values from eqs. (F.11)–(F.14) into the base case parameter vector Θ^{base}.

We now consider the higher moment $\sigma_{\nu_R^2,\nu_R} \equiv \mathrm{Cov}\left(\nu_R^2,\nu_R\right)$, and make some additional distributional assumptions that permit us to compute $\sigma_{\nu_R^2,\nu_R}$ as a function of $\overline{\nu}_R$ and $\sigma_{\nu_R}^2$ from eqs. (F.13) and (F.14). Namely, we assume now that ν_R is *lognormally distributed*, and define moments of the natural logarithm of ν_R, $\ln\left(\nu_R\right)$, as

$$\mu \equiv \text{mean of } \ln\left(\nu_R\right)$$

and

$$\sigma^2 \equiv \text{variance of } \ln\left(\nu_R\right).$$

Then, as a function of these parameters, the probability density function of ν_R, $f_{\nu_R}\left(\nu_R\right)$, is—in terms of the parameters μ and σ^2—

$$f_{\nu_R}\left(\nu_R\right)=\frac{1}{\nu_R\sigma\sqrt{2\pi}}\cdot\exp\left\{-\frac{\left[\ln\left(\nu_R\right)-\mu\right]^2}{2\sigma^2}\right\}. \tag{F.17}$$

As a function of the distributional parameters μ and σ^2 in eq. (F.17), we may show (see, e.g., Hastings and Peacock 1975) that ν_R has a mean $\overline{\nu}_R$ of

$$\overline{\nu}_R=\exp\left(\mu+\frac{\sigma^2}{2}\right), \tag{F.18}$$

a standard deviation σ_{ν_R} of

$$\sigma_{v_R} = \exp(\mu)\left[\exp\left(2\sigma^2\right) - \exp\left(\sigma^2\right)\right]^{1/2}, \quad \text{(F.19)}$$

a variance $\sigma_{v_R}^2$ of

$$\sigma_{v_R}^2 = \exp\left(2\mu + \sigma^2\right)\left[\exp\left(\sigma^2\right) - 1\right], \quad \text{(F.20)}$$

and a coefficient of skewness α_3 of

$$\alpha_3 = \left[\exp\left(\sigma^2\right) + 2\right]\left[\exp\left(\sigma^2\right) - 1\right]^{1/2}. \quad \text{(F.21)}$$

From note 217 in text chapter 6, we may express the higher moment $\sigma_{v_R^2, v_R}$ as

$$\sigma_{v_R^2, v_R} = \left(\sigma_{v_R}^2\right)^{3/2}\left[\alpha_3 + \frac{2}{V_{v_R}}\right], \quad \text{(F.22)}$$

where α_3 is the coefficient of skewness and V_{v_R} is the coefficient of variation of v_R, that is

$$V_{v_R} \equiv \frac{\sigma_{v_R}}{\bar{v}_R}. \quad \text{(F.23)}$$

To express $\sigma_{v_R^2, v_R}$ in terms of the underlying distributional parameters μ and σ^2—and ultimately in terms of $\bar{v}_R$ and $\sigma_{v_R}^2$—begin by substituting for V_{v_R} in eq. (F.22) from eq. (F.23):

$$\sigma_{v_R^2, v_R} = \left(\sigma_{v_R}^2\right)^{3/2}\left[\alpha_3 + \frac{2\bar{v}_R}{\sigma_{v_R}}\right]. \quad \text{(F.24)}$$

Next, substitute into eq. (F.24) from eqs. (F.18)–(F.21) to obtain

$$\sigma_{v_R^2,v_R} = \left\{\exp\left(2\mu+\sigma^2\right)\left[\exp\left(\sigma^2\right)-1\right]\right\}^{3/2}\left\{\left[\exp\left(\sigma^2\right)+2\right]\left[\exp\left(\sigma^2\right)-1\right]^{1/2} + \frac{2\exp\left(\mu+\frac{\sigma^2}{2}\right)}{\exp(\mu)\left[\exp\left(2\sigma^2\right)-\exp\left(\sigma^2\right)\right]^{1/2}}\right\},$$

which simplifies to

$$\sigma_{v_R^2,v_R} = \exp\left(3\mu+\frac{5}{2}\sigma^2\right)\left[\exp\left(2\sigma^2\right)-1\right]^2. \tag{F.25}$$

Solving eqs. (F.18) and (F.20) for μ and σ^2 yields

$$\mu = \ln\left(\bar{v}_R\right) - \frac{1}{2}\ln\left(\frac{\sigma_{v_R}^2}{\bar{v}_R^2}+1\right) \tag{F.26}$$

and

$$\sigma^2 = \ln\left(\frac{\sigma_{v_R}^2}{\bar{v}_R^2}+1\right). \tag{F.27}$$

Substituting eqs. (F.26) and (F.27) into eq. (F.25), we have

$$\sigma_{v_R^2,v_R} = \exp\left\{3\left[\ln\left(\bar{v}_R\right)-\frac{1}{2}\ln\left(\frac{\sigma_{v_R}^2}{\bar{v}_R^2}+1\right)\right]+\frac{5}{2}\ln\left(\frac{\sigma_{v_R}^2}{\bar{v}_R^2}+1\right)\right\}\left\{\exp\left[2\ln\left(\frac{\sigma_{v_R}^2}{\bar{v}_R^2}+1\right)\right]-1\right\}^2,$$

which we may simplify as[388]

$$\sigma_{v_R^2,v_R} = \sigma_{v_R}^3\left(\frac{\sigma_{v_R}}{\bar{v}_R}\right)\left(\frac{\sigma_{v_R}^2}{\bar{v}_R^2}+1\right)\left(\frac{\sigma_{v_R}^2}{\bar{v}_R^2}+2\right)^2 > 0. \tag{F.28}$$

[388] The positivity of $\sigma_{v_R^2,v_R}$ follows from $\bar{v}_R > 0$, recalling that v_R is lognormally distributed.

In terms of the coefficient of variation V_{v_R} (see eq. (F.23)), we may write eq. (F.28) as

$$\sigma_{v_R^2, v_R} = \sigma_{v_R}^3 V_{v_R} \left(V_{v_R}^2 + 1 \right) \left(V_{v_R}^2 + 2 \right)^2 > 0 . \tag{F.29}$$

Using values of the mean and standard deviation of v_R from eqs. (F.13) and (F.16), we may compute V_{v_R} from eq. (F.23) as

$$V_{v_R} \equiv \frac{\sigma_{v_R}}{\bar{v}_R} = \frac{242.08 \text{ MWh}}{334.60 \text{ MWh}} = 0.72349 . \tag{F.30}$$

Substituting from eqs. (F.16) and (F.30), eq. (F.29) becomes (to five significant figures)

$$\sigma_{v_R^2, v_R} = \left(242.08 \text{ MWh} \right)^3 \left(0.72349 \right) \left(0.72349^2 + 1 \right) \left(0.72349^2 + 2 \right)^2 ,$$

or

$$\sigma_{v_R^2, v_R} = 9.9568\text{e}7 \left(\text{MWh} \right)^3 . \tag{F.31}$$

F.2 Forward market

Figure F.2 below depicts a representative forward market demand function $D^f \left(p^f , \varepsilon_0^f \right)$. The figure also depicts an empirical reference price $p_{empir}^{f,\,mean}$ and empirical (aggregate) reference quantity $q_{empir}^{f,\,mean}$. Subsection F.2.1 below provides a value of $p_{empir}^{f,\,mean}$ based on the literature, and explains how we compute $q_{empir}^{f,\,mean}$. Unless otherwise noted, chapter 7's numerical analysis considers forward market SFs over the range

$$p^f \in \left[0,\ 2{,}750 \right] \text{ \$/MWh} . \tag{F.32}$$

This price range in (F.32) also includes the price $\$2{,}500/\text{MWh}$, which was the applicable (software-imposed) California PX price cap as of March 1999 (Market Monitoring Committee of the California Power Exchange 1999, 47).[389]

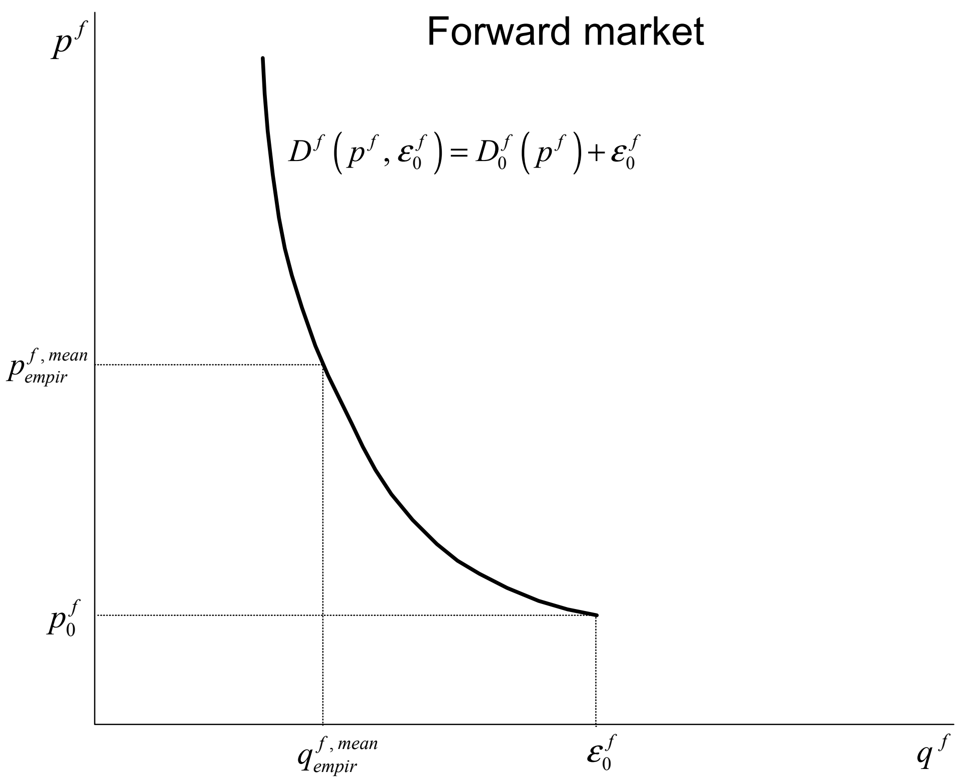

FIGURE F.2: FORWARD MARKET GEOMETRY

F.2.1 Prices and quantities

The following parameters are available directly from the literature:

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[389] We emphasize that the restriction in the range of forward market prices considered, $p^f \in [0,\ 2{,}750]\ \$/\text{MWh}$, is for computational purposes only; it applies, in particular, to the specific numerical examples of text section 7.6. Text subsection 3.1.5's definition of $\overline{S}_i^f\left(p^f\right)$ as a function over $p^f \in \mathbb{R}$ still applies. If desired, we may specify the interval of p^f over which we compute the functions $\overline{S}_i^f\left(p^f\right)$ to include negative prices.

- $p^{f,mean}_{empir} = \$26.60/\text{MWh}$: Mean California PX unconstrained forward market price during the reference period, averaged over all hours (Borenstein, Bushnell and Wolak 2002, 1393)
- $q^{f,mean}_{empir,tot} = 21{,}579\text{ MWh}$: Mean aggregate forward market demand in the California PX (facing *all* generating units) during the period April 1998 to April 1999,[390] averaged over all hours (California Power Exchange 1999, 27)

We now compute the empirical (aggregate) reference quantity $q^{f,mean}_{empir}$ for the forward market, corresponding to the (aggregate) reference demand level facing only fossil-fired generating units. Recalling the mean hourly spot market demand $q^{s,mean}_{empir} = 4{,}955\text{ MWh}$, assume that the fraction of this quantity that is transacted in the forward market is given by the ratio $q^{f,mean}_{empir,tot} \big/ q^{s,mean}_{empir,tot}$. We may then compute $q^{f,mean}_{empir}$ as

$$q^{f,mean}_{empir} = q^{s,mean}_{empir} \cdot \frac{q^{f,mean}_{empir,tot}}{q^{s,mean}_{empir,tot}} = \left(4{,}955\text{ MWh}\right) \cdot \frac{21{,}579\text{ MWh}}{26{,}511\text{ MWh}} = 4{,}033\text{ MWh}. \quad \text{(F.33)}$$

F.2.2 Consumers' risk preferences

Text subsection 6.2.1 defined the parameter λ_j as the constant absolute risk aversion—or CARA—coefficient for consumer *j*. The purpose of this subsection is to determine an appropriate value of the CARA coefficient, λ_R, for the representative consumer *R* introduced later in text chapter 6. In the absence of data on consumers' risk aversion in the context of electricity markets, we turn to other economic settings to provide a basis

[390] This period does not coincide exactly with the reference period, but we take this as a suitable approximation of average forward market demand during the reference period.

for quantitative estimates of λ_R. It is reasonable to suppose that consumers' risk preferences in electricity markets are comparable to those governing behavior in markets for other goods and services. Accordingly, this subsection briefly surveys the literature on empirical estimates of the CARA coefficient from a variety of economic contexts, and places these estimates on a comparable basis.

Table F.1 below reports the results of numerous empirical studies of agents' risk preferences, conducted in a wide variety of economic settings (notably, the agricultural sector, which has often been studied in this context). The rightmost column of the table gives the estimates of the CARA coefficient λ computed in each study, expressed in uniform units of $(\$_{1999})^{-1}$ for purposes of comparison across the studies.[391] Table F.1 lists the various studies in order of increasing CARA coefficients (i.e., increasing risk aversion).[392] Note 198 in the text provides an intuitive interpretation of the CARA coefficient λ.

[391] See the notes to Table F.1 for details of currency and current-to-constant dollar conversions, where applicable.

[392] When Table F.1 reports a range of values for λ (see the respective original studies for details), the geometric mean of the endpoints of this range is used to order the studies.

TABLE F.1: EMPIRICAL ESTIMATES OF THE CONSTANT ABSOLUTE RISK AVERSION (CARA) COEFFICIENT λ (IN ORDER OF INCREASING λ)

Citation	*Object of study and data reference year*[a]	λ *(original units)*	*Constant dollar and currency conversion factors*[b]	λ $(\$_{1999})^{-1}$
Buccola (1982)	Processing tomato producer in California, USA – 1979	0.0012–0.00196 $(\$1000)^{-1}$	$PPF_{1979} = 66$	6.9e-7 – 1.12e-6
Bar-Shira, Just, and Zilberman (1997)	Farmers in Arava region, Israel – 1978	4.5e-6 $\$^{-1}$	$PPF_{1978} = 58$	2.3e-6
Lien (2002)	Lowland crop and livestock farmers in Norway – 1996	1.44e-6 NOK^{-1}	$PPF_{1996} = 115$ $9.305\ (NOK/\$)_{1996}$	1.34e-5
Ozanne (1998)	Crop and livestock farmers in the USA – 1964	4.28e-4 $\$^{-1}$	$PPF_{1964} = 23.8$	8.86e-5
Zacharias and Grube (1984)	Experiment at Agronomy South Farm, Urbana, IL, USA – 1971	9.2e-5 – 3.5e-3 $\$^{-1}$	$PPF_{1971} = 30.7$	2.5e-5 – 9.3e-4
Simmons and Pomareda (1975)	Crop farmers in Mexico exporting to the USA – 1972	0.5 $Pesos^{-1}$	$PPF_{1972} = 32.9$ $0.00632\ (Pesos/\$)_{1972}$	0.0009
Kramer and Pope (1981)	Field crop farmers in Kern County, CA, USA – 1974	0.00125–0.03 $\$^{-1}$	$PPF_{1974} = 42.9$	4.66e-4 – 1e-2
Love and Buccola (1991)	Corn and soybean farmers in Iowa, USA – 1967	0.016–0.538 $\$^{-1}$	$PPF_{1967} = 26$	0.0036–0.122
Brink and McCarl (1978)	Large corn belt cash grain farmers in the USA – 1975	0.23 $\$^{-1}$	$PPF_{1975} = 47$	0.094
Beetsma and Schotman (2001)	Television game show contestants in the Netherlands – 1996	0.11–0.24 $Guilders^{-1}$	$CPI\text{-}U_{1996} = 156.9$[c] $CPI\text{-}U_{1999} = 166.6$ $2.09\ (Guilders/\$)_{1996}$	0.22–0.47
Wolf and Pohlman (1983)	A dealer in USA Treasury Bill auctions – 1977	2–4.5 $\$^{-1}$	4.4[d]	0.5–1.0
Chavas and Holt (1996)	Corn and soybean farmers in the USA – 1967	12.171 $\$^{-1}$	$PPF_{1967} = 26$	2.8
Antle (1987)	Rice farmers in Aurepalle Village, India – 1979	3.272 $Rupees^{-1}$	$PPF_{1979} = 66$ $3.162\ (Rupees/\$)_{1979}$	5.9

Notes:

[a] Where data are drawn over multiple years, we use the midpoint of this time interval as the data reference year.

Notes to Table F.1 (cont'd):

[b] For agricultural studies, we convert dollars from the data reference year to 1999 dollars using the price indices for prices paid by farmers for all commodities, services, interest, taxes, and wage rates for the relevant years (*Economic Report of the President* 1991, Table B-98 for data reference years prior to 1975; *Economic Report of the President* 2003, Table B-101 for data reference years 1975 and later). For year t, we denote this price index as "PPF_t," and note that it is normalized using $\mathrm{PPF}_{1991} = 100$. For 1999, recalling that the CARA coefficients reported in the rightmost column of Table F.1 have units of $(\$_{1999})^{-1}$, we have $\mathrm{PPF}_{1999} = 115$. For non-agricultural studies, we report the appropriate conversion factors in the fourth column of Table F.1. We use purchasing power parity exchange rates from the Penn World Table (Heston, Summers and Aten 2002) to perform currency conversions to current dollars in the data reference year. See below for an example of the use of the various conversion factors.

[c] CPI-U_t is the consumer price index for all items in year t, where $\text{CPI-U}_{1983\text{-}1984} = 100$ (*Economic Report of the President* 2003, Table B-60).

[d] Because this particular study addresses the behavior of a Treasury bill dealer, the conversion factor 4.4 is the approximate return on \$1 invested at the average annual Treasury bill rate beginning in 1977, and compounded annually until 1999 (International Monetary Fund 2003, Treasury Bill Rate).

As an example of the conversions used in Table F.1, consider Simmons and Pomareda's 1975 study of Mexican farmers that export to the USA. The authors report a value of $\lambda = 0.5\left(\mathrm{Pesos}_{1972}\right)^{-1}$, which we convert to units of $(\$_{1999})^{-1}$ as follows:

$$0.5\left(\mathrm{Pesos}_{1972}\right)^{-1} \cdot \frac{0.00632\ \mathrm{Pesos}_{1972}}{\$_{1972}} \cdot \frac{32.9\ \mathrm{PPF}_{1972}}{115\ \mathrm{PPF}_{1999}} = 9\text{e-}4\left(\$_{1999}\right)^{-1}. \tag{F.34}$$

The CARA coefficients λ in Table F.1 (in the rightmost column) lie in the interval $[6.9\text{e-}7, 5.9]$, a range of nearly 7 orders of magnitude, with geometric mean

$$\bar{\lambda}_{geom} = 0.0033\left(\$_{1999}\right)^{-1}. \tag{F.35}$$

The economic agents whose risk preferences are characterized in Table F.1 tend to be smaller-scale (in terms of revenues, for example) and, plausibly, less financially sophisticated than most of the electricity consumers participating in the California PX. Thus, we would expect the electricity consumers that we wish to model here to be *less risk averse*, on average, than the "average" agent characterized in Table F.1. Taking the geometric mean $\bar{\lambda}_{geom}$ of the coefficients λ in Table F.1 to be representative of the agents characterized in the table, the above observation suggests that the representative consumer's CARA coefficient λ_R is related to $\bar{\lambda}_{geom}$ as follows:

$$\lambda_R < \bar{\lambda}_{geom} = 0.0033\left(\$_{1999}\right)^{-1}. \tag{F.36}$$

Based on the considerable range of CARA coefficients reported in Table F.1, we assume, more specifically, that λ_R and $\bar{\lambda}_{geom}$ differ by *one order of magnitude*. We assume, therefore, that as a rough estimate $\left(\lambda_R\right)^{est}$ of λ_R, we may use the value

$$\left(\lambda_R\right)^{est} = 0.00033\left(\$_{1999}\right)^{-1}. \tag{F.37}$$

Because of the approximate nature of the above discussion, we use eq. (F.37) as merely an *initial condition* for λ_R in text section 7.5's benchmarking procedure using the discrete Excel model. That is, we use the value $\left(\lambda_R\right)^{est}$ in eq. (F.37) as an element of the parameter vector $\Theta^{(0)}$ in text problem (7.55), and hence also in $\Theta^{(1)}$, the vector of initial values for text problem (7.56). The solution to this benchmarking problem (step 2 of the benchmarking procedure) yields an *endogenous* value of λ_R—only slightly different from $\left(\lambda_R\right)^{est}$ above—namely,

$$\left(\lambda_R\right)^{opt} = 0.00031984\left(\$_{1999}\right)^{-1}. \tag{F.38}$$

Recalling the illustrative interpretation of the CARA coefficient from note 198 in chapter 6, we observe that the value of $\left(\lambda_R\right)^{opt}$ in eq. (F.38) corresponds to a "risk tolerance" parameter τ_R of

$$\tau_R \equiv \frac{1}{\lambda_R} = \frac{1}{0.00031984\,(\$_{1999})^{-1}} = (\$_{1999})3126.60\,.$$

For a risk-averse consumer R with CARA coefficient λ_R given in eq. (F.38), the interpretation of the risk tolerance τ_R is that the consumer R is (approximately) indifferent between accepting and not accepting a lottery offering even odds over payoffs of $\tau_R = (\$_{1999})3126.60$ and $-\tau_R/2 = -(\$_{1999})1563.30$. We incorporate the value $(\lambda_R)^{opt}$ from eq. (F.38) into the base case parameter vector Θ^{base}.

F.3 Summary

Collecting the numerical results documented in this appendix, we may write the base case parameter vector Θ^{base} as (rounding results to three significant figures)

$$\Theta^{base} \equiv \begin{pmatrix} c_{01} \\ c_{02} \\ c_1 \\ c_2 \\ e^s_{dem} \\ \bar{\eta}_R \\ \sigma^2_{\eta_R} \\ \bar{v}_R \\ \sigma^2_{v_R} \\ \lambda_R \end{pmatrix}^{base} = \begin{pmatrix} \$25.60/\text{MWh} \\ \$30.50/\text{MWh} \\ \$0.000341/(\text{MWh})^2 \\ \$0.00326/(\text{MWh})^2 \\ -5.95\text{e-}5 \\ 4640\ \text{MWh} \\ 2.46\text{e}6\ \text{MWh}^2 \\ 335\ \text{MWh} \\ 5.86\text{e}4\ \text{MWh}^2 \\ 3.20\text{e-}4\ \$^{-1} \end{pmatrix}. \tag{F.39}$$

For ease of reference, the vector Θ^{base} in eq. (F.39) also appears in the text as eq. (7.46).

The melancholia of everything COMPLETED!

—Nietzsche, *Beyond Good and Evil*

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