



## Using Mathematical Programming for Electricity Spot Pricing

WILLIAM W. HOGAN,<sup>1</sup> E. GRANT READ<sup>2</sup> and BRENDAN J. RING<sup>2</sup>

<sup>1</sup>Harvard University, USA and <sup>2</sup>University of Canterbury, New Zealand

Recent moves around the world to introduce competition into electricity markets have created a need for mechanisms to determine electricity spot prices which provide good incentives for market coordination. Duality theory suggests that such prices can be found by solving a mathematical program. We derive implicit prices corresponding to an actual half-hourly dispatch of a full a.c. power system, and discuss the application of spot pricing in New Zealand and the United States. Copyright © 1996. Published by IFORS/Elsevier Science Ltd.

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### INTRODUCTION

Electric power systems have traditionally been operated as regulated monopolies, partly to cope with the complexity of their operation and planning. In recent years, however, there has been a widespread realisation that only the transmission of power is a natural monopoly, and that efficiency gains can be made by deregulating and fostering competition within the generation and consumption sectors.<sup>†</sup> This change of perspective is apparent from the reforms implemented in countries as diverse as Chile, the UK, Norway, Argentina, New Zealand, Australia and those proposed for California (CPUC, 1995).

This new structure has created a need for an efficient transmission pricing regime, and particularly one which addresses the externalities which power system transactions create (Hogan, 1992). Due both to the non-linearities inherent in power system operation, and the need to differentiate between physical operation and financial transactions, three basic mechanisms are required for efficient market operations, these being a short run marginal cost spot pricing regime to encourage efficient network usage, financial hedging contracts to minimise exposure to price volatility, and a mechanism for network fixed cost recovery, which, in conjunction with the above, creates efficient long run investment signals, while minimising distortion to short run operations. (Hogan, 1992; Read and Ring, 1996). This paper focuses on the basic procedures for deriving spot prices.

While simple implementations of spot pricing have been in use in the electricity industry for 40 years, Caramanis *et al.* (1982) were the first to derive spot prices which vary across space to reflect the marginal cost of losses and the costs of network congestion, and across time in response to changing demand and generator availability, with each generator or consumer simply selling or buying energy at the local spot price. In Schweppe *et al.* (1988) these prices were to be determined by forming the dual mathematical programming problem corresponding to a direct current (d.c.) approximation to the power system dispatching problem. Baughman and Siddiqi (1991) and Hogan (1993) have since demonstrated the importance of modelling the full complexity of an alternating current (a.c.) power system if pricing consistency and accuracy is to be achieved.

In this paper we examine an extension of the methodology of Caramanis *et al.* (1982) and Schweppe *et al.* (1988) to account for the complexities of a.c. systems within a framework that could be integrated with a real dispatch system. Our derivation is a simplified version of those presented by Hogan (1991), Read and Ring (1995), and Ring (1995). We discuss the application of spot pricing models in New Zealand and the US.

The dispatch problem involves complexity that goes beyond the simplification of formal models of optimal power flows. To avoid the necessity of changing the way in which the actual dispatch is

<sup>†</sup> Here we make no distinction between transmission and distribution, using only the generic term transmission to refer to the wires component of the electricity sector.

Correspondence: E. Grant Read, Department of Management, University of Canterbury, Private Bag 4800, Christchurch, New Zealand



formed, we take an observed power system dispatch to be the optimal solution based on the preferences or bids of the market participants. Given this dispatch as determined by the system operator, we linearise a formal, albeit simplified, dispatch formulation about the observed solution, and hence produce a linear primal problem from which dual prices can readily be determined that capture the main structure of the dispatch decisions. This approach, which we refer to as *ex post* pricing, means that prices, and hence financial transactions, are based on what actually happened. This simplifies the determination of the prices. As long as the dispatch is consistent with pre-dispatch (*ex ante*) offers and bids, 'dual' mathematical programming techniques can be used to find prices which are consistent with both the dispatch and the bids and offers upon which it is based, and which must therefore satisfy all market participants. Note that this *ex post* approach does not preclude the use of *ex ante* price signalling mechanisms to coordinate the dispatch, but *ex ante* prices should be interpreted as forecasts of, or hedges against, *ex post* prices.

The work reported here represents a fundamental change in emphasis from the 'traditional' role of Operations Research in centralised planning. Rather than attempting to determine the optimal operation and evolution of an electricity system, the problem separates into two components. The system operator determines the optimal short run (primal) dispatch based on the bids, and the *ex post* (dual) prices derived in this work provide the appropriate signals upon which individual decision makers in the market can make long run decisions. This approach allows market participants to take better advantage of their 'individual' knowledge and to innovate, while giving them strong incentives to act in ways which contribute value to the system as a whole. This work highlights the contribution which mathematical programming, and more particularly duality theory, can make, not only to the analysis of policy options but to supporting markets by providing incentives which are finely balanced so as to simultaneously achieve the benefits of both competition and coordinate, without requiring the debilitating impact of organisational centralisation. Read (1996) suggests that this approach represents part of a general trend in Operations Research modelling, at least in the electricity sector, from primal to dual optimisation as more reliance is placed on market and incentive mechanisms, and less on top-down planning, in many economies and organisations around the world.

In the following section we describe the primal dispatch problem, and discuss its linearisation about an observed, and assumed to be optimal, solution. This is followed by the presentation of the dual pricing problem. We subsequently simplify the dual and discuss the implications of mathematic programming theory to its use and interpretation. We do not extensively discuss the economic and policy implications of this model as these are discussed by Hogan (1992) and Read and Ring (1995, 1996). A brief summary is then presented of a number of issues raised in applications of this type of model, both in New Zealand and the US.

## THE PRIMAL DISPATCH PROBLEM

Power system networks behave very differently from 'conventional' Operations Research networks. In particular, rather than being viewed as a series of interconnected paths allowing identifiable units of a commodity to flow between two points along a particular path, power system networks must be viewed more as 'pools' into which identifiable units of a commodity are introduced, and from which identifiable units are taken. It is not physically meaningful to think of a unique supplier of a given physical unit of power removed from the pool. Power system networks are also fundamentally different in that the flows which do occur, do so in accordance with the laws of physics, rather than the desires of some decision maker. In particular it is not possible to choose how flows will be split between alternative paths, as in a conventional 'network flow' formulation, and this leads to a fundamentally different formulation in which, for example, it is quite possible for flows between buses to be constrained by a single congested line even though parallel, and apparently unconstrained, paths exist.

The pool nature of power systems can only be fully represented using complex engineering equations. Ultimately, however, there is a limit as to the detail with which power system pricing models can be taken, as it is not practical, or even possible, to model, optimise, or even monitor, every last facet of power system operation. Hence, we must accept that some boundaries must be placed on the scope of the transmission pricing problem. This should not have a serious impact provided these



boundaries are consistent with the market's level of sophistication. An appropriate boundary seems to be that which separates the longer term integer generator capacity problem, known as the Unit Commitment problem, from the shorter term problem of determining actual operating levels. For the purposes of pricing it is convenient to assume that an Optimal Power Flow (OPF) formulation (Heneault and Galiana, 1991) is used to represent the short term problem. The dispatch adjusts continuously, but for pricing purposes we may treat the dispatch as constant over a short period of time, say a half hour, using the available generator capacities determined by the unit commitment problem to determine the optimal operating level of each generator (and potentially load), as well as the transmission line power flows, while satisfying a range of system constraints. In practice most system controllers use less formal approaches, but their goal is still the same, and that is all that is required for the pricing policy discussed here to be relevant.

### System representation

The key electrical variables of an OPF require some explaining. These variables are voltage and power. Other variables, such as tap changing transformer ratios, can be included (Read *et al.*, 1995) but are omitted here. Both voltage and power are complex variables, with voltage comprising, in polar coordinates, a voltage magnitude and phase angle, while power comprises, in cartesian coordinates, active and reactive power (Wood and Wollenberg, 1984). Active power is the 'real' component of power with which most people are familiar, while reactive power is the 'imaginary' component which plays an important role in the transmission of power, and is every bit as real (in a physical sense) as active power.

Ohm's law describes the functional relationship between (complex) voltage and (complex) power and consequently all power injections can be described as being dependent on the voltages, or vice versa. In practice, however, it is more convenient to define a subset of the voltages and power injections as being the independent variables, with all other terms dependent upon them. While Read and Ring (1995) use a general representation of the independent variables, for ease of discussion we assume that the independent variables are the active and reactive power net injections at each bus (or node) in the network except for one, as in Hogan (1991). The requirement that power be conserved means that both active and reactive power at one bus, referred to as the *swing bus*, must be free to vary so as to match generation with demand and power losses. At the swing bus complex voltage is independent while active and reactive power are dependent variables. The swing bus used for pricing purposes only fills this role notionally, and hence is arbitrary, with real time variations in load actually being met by the power sources (either generation or load curtailment) which can respond in the most economical manner on the margin. These buses are referred to as *marginal buses*.

The relationship between complex power and complex voltage involves the phase angle differences between buses. Thus one phase angle value, which we take to be that at the swing bus, is an arbitrary reference value and has no direct consequences to the optimisation, and hence can be ignored. Further, the voltage magnitude at the swing bus can, for the purposes of discussion, be treated as an exogenous variable and therefore be treated as a constant. Thus the independent variables we model are the active and reactive power injections at all non-swing buses, and we refer to this as a *PQ* representation.

The swing bus is denoted by the index  $s$ , or when included in a set by the suffix  $S$ . We use  $PX$  to denote the set of all buses other than the swing bus, and  $PXS$  to be the set of all buses. Thus, for example, although individual buses are indicated by subscripts,  $P^{PXS}$  represents the vector of active power net injections for all buses in the system.

Our OPF objective function has the form:

$$\text{Minimise Cost}(P_G^{PXS}, Q_G^{PXS}) \quad (1)$$

This equation states that the aim of the dispatcher is to minimise the total cost of generating active power  $P_G^{PXS}$  and reactive power  $Q_G^{PXS}$ , at each bus where  $\text{Cost}(P_G^{PXS}, Q_G^{PXS})$  describes the total fuel cost of generation. For the present discussion, this objective is assumed to be convex, but may be non-differentiable at some points. In reality, these functions may be non-convex, though separable with respect to each generators' output. In practice, however, a piece-wise quadratic approximation is generally used, with smoothed transition across the non-differentiable points (Bacher, 1992).



There will often be no direct fuel cost associated with reactive power production, hence, while reactive power has been included for generality, the objective function may actually only involve active power generation costs. Without loss of generality, load benefits could be included to emphasise a welfare maximising objective appropriate from an economic perspective (Hogan, 1991). The cost minimising objective is preserved here as more intuitive from a traditional power system viewpoint.

In minimising the objective function it is necessary to ensure that power, both active and reactive, is conserved, which is a fundamental physical constraint. The relevant constraints are:

$$\sum_{i \in PXS} (P_{Gi} - P_{Di}) - L_P(\mathbf{P}_G^{PX} - \mathbf{P}_D^{PX}, \mathbf{Q}_G^{PX} - \mathbf{Q}_D^{PX}) = 0 \quad (2)$$

$$\sum_{i \in PXS} (Q_{Gi} - Q_{Di}) - L_Q(\mathbf{P}_G^{PX} - \mathbf{P}_D^{PX}, \mathbf{Q}_G^{PX} - \mathbf{Q}_D^{PX}) = 0 \quad (3)$$

In these equations losses, of both active ( $L_P$ ) and reactive ( $L_Q$ ) power, are defined as functions of the independent variables, where we have represented the net power injections as the difference between generation and demand vectors, demand being represented by  $\mathbf{P}_D^{PX}$  and  $\mathbf{Q}_D^{PX}$ . Active power losses are caused by energy escaping from transmission lines in the form of heat. Reactive power may be consumed or produced by the transmission system. Equations (2) and (3) state, for active and reactive power respectively, that total production less consumption by customers must equal losses. Simplistically, if inadequate generation is available to satisfy these relationships at the systems nominal power frequency, then the frequency will drop, causing the generators to produce more power, hence satisfying the constraints. Such frequency excursions can damage equipment, and should be avoided. We here assume a 'snapshot' view of the system, and hence do not explicitly model frequency constraints.

Equations (2) and (3) effectively define the dependent power injections of the swing bus, as the losses are fixed given the value of the independent variables, as are all non-swing bus injections. These equations can, therefore, be stated more simply in the form of (4) and (5), where  $P_s$  and  $Q_s$ , are respectively, the net swing bus active and reactive power injections.

$$-P_s(\mathbf{P}_G^{PX} - \mathbf{P}_D^{PX}, \mathbf{Q}_G^{PX} - \mathbf{Q}_D^{PX}) + (\mathbf{P}_{Gs} - \mathbf{P}_{Ds}) = 0 \quad (4)$$

$$-Q_s(\mathbf{P}_G^{PX} - \mathbf{P}_D^{PX}, \mathbf{Q}_G^{PX} - \mathbf{Q}_D^{PX}) + (\mathbf{Q}_{Gs} - \mathbf{Q}_{Ds}) = 0 \quad (5)$$

While the equality constraints make (4) and (5) appear non-convex, this can be overcome for *ex post* pricing purposes by recognising that the OPF is solved with a fixed set of generators dispatched, and with at least one being a slack variable, effectively allowing these constraints to be treated as inequalities (Glavitsch, 1992).

The dependent voltage magnitudes,  $V$ , at the  $PX$  buses can be defined in a similar manner to the swing bus injections, as shown in (6).

$$-V_n(\mathbf{P}_G^{PX} - \mathbf{P}_D^{PX}, \mathbf{Q}_G^{PX} - \mathbf{Q}_D^{PX}) + V_n = 0 \quad \forall n \in PX \quad (6)$$

Likewise, the average active and reactive power flows,  $\bar{P}$  and  $\bar{Q}$  respectively, on each of the  $K$  transmission lines, can be defined by (7) and (8).

$$-\bar{P}_k(\mathbf{P}_G^{PX} - \mathbf{P}_D^{PX}, \mathbf{Q}_G^{PX} - \mathbf{Q}_D^{PX}) + \bar{P}_k = 0 \quad \forall k \in K \quad (7)$$

$$-\bar{Q}_k(\mathbf{P}_G^{PX} - \mathbf{P}_D^{PX}, \mathbf{Q}_G^{PX} - \mathbf{Q}_D^{PX}) + \bar{Q}_k = 0 \quad \forall k \in K \quad (8)$$

Unlike (4) and (5), there is no certainty that (6), (7), and (8) are convex, although our experience suggests that local non-convexity is only an issue in lightly loaded power systems. The implication of such non-convexities is that the observed power system solution, which we assume to be globally optimal, may in fact be only locally optimal. While this may have some revenue implications for the system operator, which collects economic rents from the dispatch, these effects can be accommodated by a relatively small adjustment to the fixed cost recovery mechanisms. The disposition of economic rents and other revenue issues are beyond the scope of this paper, but will be important as a policy matter to keep the correct incentives for the system operator (Hogan, 1992; San Diego, 1995). Other



than this, we conjecture that under normal operating conditions the impact of non-convexities even in this simplified OPF problem will be minimal.†

Bounds must be imposed to define the feasible range of the dispatch variables:

$$V_n^{\min} \leq V_n \leq V_n^{\max} \quad \forall n \in PX \quad (9)$$

$$\bar{P}_k^{\min} \leq \bar{P}_k \leq \bar{P}_k^{\max} \quad \forall k \in K \quad (10)$$

$$\bar{Q}_k^{\min} \leq \bar{Q}_k \leq \bar{Q}_k^{\max} \quad \forall k \in K \quad (11)$$

$$P_{Gi}^{\min} \leq P_{Gi} \leq P_{Gi}^{\max} \quad \forall i \in PXS \quad (12)$$

$$Q_{Gi}^{\min} \leq Q_{Gi} \leq Q_{Gi}^{\max} \quad \forall i \in PXS \quad (13)$$

$$P_{Di} = P_{Di}^{\text{set}} \quad \forall i \in PXS \quad (14)$$

$$Q_{Di} = Q_{Di}^{\text{set}} \quad \forall i \in PXS \quad (15)$$

Equations (14) and (15) have set the upper and lower bounds for active and reactive power demand at each bus to be identical, i.e., the demands are set externally to the dispatch problem. For an optimal observed setting of these values, therefore, the shadow prices on these bounds should equal the prices determined by a welfare maximising objective when demand is not fixed.

### Linearisation of the OPF

Equations (1) and (4)–(15) describe a simple representation of an Optimal Power Flow problem. We do not assume that this particular optimisation problem was formally solved to determine the dispatch, but using the *ex post* philosophy discussed above, we can still linearise this primal formulation about an observed dispatch. As well as simplifying the modelling of a full a.c. power system, this approach results in prices which correspond to the actual dispatch, rather than to what was expected prior to the dispatch.

The linearisation is performed using a standard first order Taylor's expansion of the non-linear terms, that is:

$$f(x) \approx f(x^*) + \frac{\partial f}{\partial x}(x - x^*) \quad (16)$$

Here  $x^*$  denotes the observed value of a variable  $x$ . We assume  $x^*$  to be optimal. Due to the possibility of discontinuous first derivatives of the objective function we must distinguish between increasing active or reactive power generation beyond their observed values, or reducing them below their observed values. Thus we define:

$$P_{Gi} = P_{Gi}^* + P_{Gi}^+ - P_{Gi}^- \quad (17)$$

$$Q_{Gi} = Q_{Gi}^* + Q_{Gi}^+ - Q_{Gi}^- \quad (18)$$

$$P_{Gi}^+, P_{Gi}^-, Q_{Gi}^+, Q_{Gi}^- \geq 0 \quad (19)$$

Here  $P_{Gi}^+$ , an increase in active power generation, has a marginal fuel cost of  $c_{Pi}^+$ , which exceeds  $c_{Pi}^-$ , the marginal reduction in fuel cost associated with  $P_{Gi}^-$ , a decrease in active power generation. Analogous definitions, though in terms of reactive power generation, apply for  $c_{Qi}^+$ ,  $c_{Qi}^-$ ,  $Q_{Gi}^+$  and  $Q_{Gi}^-$ .

Using (17), (18) and (19), the canonical form of the linearised OPF is that described by (20)–(37). In each of (32)–(35) one of the 'change in generation' terms of (17) and (18) is a slack variable, and has been dropped from the equation. Constants in the objective function have been ignored while those in the constraints have been moved to the right hand side. Functional forms for the constants in the constraints are given in the Appendix. The shadow price associated with each constraint is given on the right.

†For example, (6) may be strictly concave. For the upper bounds on voltage this would violate the convexity condition. However, the upper bounds typically apply in lightly loaded conditions, which are not of interest in pricing when there is not likely to be out-of-merit-order dispatch. And for heavily loaded conditions, the lower bounds on voltage and a concave voltage function would be consistent with the convexity conditions.



$$\begin{aligned} & \text{Minimise} \\ & \sum_{i \in PXS} (c_{Pi}^+ P_{Gi}^+ - c_{Pi}^- P_{Gi}^-) + \sum_{i \in PXS} (c_{Qi}^+ Q_{Gi}^+ - c_{Qi}^- Q_{Gi}^-) \end{aligned} \quad (20)$$

subject to:

$$-\sum_{i \in PX} \frac{\partial P_s}{\partial P_i} (P_{Gi}^+ - P_{Gi}^- - P_{Di}) - \sum_{i \in PX} \frac{\partial P_s}{\partial Q_i} (Q_{Gi}^+ - Q_{Gi}^- - Q_{Di}) + (P_{Gs}^+ - P_{Gs}^- - P_{Ds}) = A_1^* : \lambda_P \quad (21)$$

$$-\sum_{i \in PX} \frac{\partial Q_s}{\partial P_i} (P_{Gi}^+ - P_{Gi}^- - P_{Di}) - \sum_{i \in PX} \frac{\partial Q_s}{\partial Q_i} (Q_{Gi}^+ - Q_{Gi}^- - Q_{Di}) + (Q_{Gs}^+ - Q_{Gs}^- - Q_{Ds}) = A_2^* : \lambda_Q \quad (22)$$

$$-\sum_{i \in PX} \frac{\partial V_n}{\partial P_i} (P_{Gi}^+ - P_{Gi}^- - P_{Di}) - \sum_{i \in PX} \frac{\partial V_n}{\partial Q_i} (Q_{Gi}^+ - Q_{Gi}^- - Q_{Di}) + V_n = A_{3n}^* : \mu_n \quad \forall n \in PX \quad (23)$$

$$-\sum_{i \in PX} \frac{\partial \bar{P}_k}{\partial P_i} (P_{Gi}^+ - P_{Gi}^- - P_{Di}) - \sum_{i \in PX} \frac{\partial \bar{P}_k}{\partial Q_i} (Q_{Gi}^+ - Q_{Gi}^- - Q_{Di}) + \bar{P}_k = A_{4k}^* : \eta_{Pk} \quad \forall k \in K \quad (24)$$

$$-\sum_{i \in PX} \frac{\partial \bar{Q}_k}{\partial P_i} (P_{Gi}^+ - P_{Gi}^- - P_{Di}) - \sum_{i \in PX} \frac{\partial \bar{Q}_k}{\partial Q_i} (Q_{Gi}^+ - Q_{Gi}^- - Q_{Di}) + \bar{Q}_k = A_{5k}^* : \eta_{Qk} \quad \forall k \in K \quad (25)$$

$$-\bar{P}_k \geq -\bar{P}_k^{\max} : v_{\bar{P}_k}^+ \quad \forall k \in K \quad (26)$$

$$\bar{P}_k \geq \bar{P}_k^{\min} : v_{\bar{P}_k}^- \quad \forall k \in K \quad (27)$$

$$-\bar{Q}_k \geq -\bar{Q}_k^{\max} : v_{\bar{Q}_k}^+ \quad \forall k \in K \quad (28)$$

$$\bar{Q}_k \geq \bar{Q}_k^{\min} : v_{\bar{Q}_k}^- \quad \forall k \in K \quad (29)$$

$$-V_n \geq -V_n^{\max} : v_{V_n}^+ \quad \forall n \in PX \quad (30)$$

$$V_n \geq V_n^{\min} : v_{V_n}^- \quad \forall n \in PX \quad (31)$$

$$-P_{Gi}^+ \geq -P_{Gi}^{\max} + P_{Gi}^* : v_{Pi}^+ \quad \forall i \in PXS \quad (32)$$

$$-P_{Gi}^- \geq P_{Gi}^{\min} - P_{Gi}^* : v_{Pi}^- \quad \forall i \in PXS \quad (33)$$

$$-Q_{Gi}^+ \geq -Q_{Gi}^{\max} + Q_{Gi}^* : v_{Qi}^+ \quad \forall i \in PXS \quad (34)$$

$$-Q_{Gi}^- \geq Q_{Gi}^{\min} - Q_{Gi}^* : v_{Qi}^- \quad \forall i \in PXS \quad (35)$$

$$P_{Di} = P_{Di}^* : \beta_{Pi} \quad \forall i \in PXS \quad (36)$$

$$Q_{Di} = Q_{Di}^* : \beta_{Qi} \quad \forall i \in PXS \quad (37)$$

### THE DUAL PROBLEM

The dual mathematical programming problem corresponding to the linearised OPF can be formed in the usual manner, producing (38)–(47). The primal variables corresponding to each dual constraint are shown on the right.

$$\begin{aligned} & \text{Maximise} \\ & \lambda_P A_1^* + \lambda_Q A_2^* + \sum_{n \in PX} \mu_n A_{3n}^* + \sum_{k \in K} \eta_{Pk} A_{4k}^* + \sum_{k \in K} \eta_{Qk} A_{5k}^* + \sum_{k \in K} (-v_{\bar{P}_k}^+ \bar{P}_k^{\max} + v_{\bar{P}_k}^- \bar{P}_k^{\min}) \\ & + \sum_{k \in K} (-v_{\bar{Q}_k}^+ \bar{Q}_k^{\max} + v_{\bar{Q}_k}^- \bar{Q}_k^{\min}) + \sum_{n \in PX} (-v_{V_n}^+ V_n^{\max} + v_{V_n}^- V_n^{\min}) + \sum_{i \in PXS} \beta_{Pi} P_{Di}^* + \sum_{i \in PXS} \beta_{Qi} Q_{Di}^* \\ & + \sum_{i \in PXS} (v_{Pi}^+ (-P_{Gi}^{\max} + P_{Gi}^*) + v_{Pi}^- (P_{Gi}^{\min} - P_{Gi}^*)) + \sum_{i \in PXS} (v_{Qi}^+ (-Q_{Gi}^{\max} + Q_{Gi}^*) + v_{Qi}^- (Q_{Gi}^{\min} - Q_{Gi}^*)) \end{aligned} \quad (38)$$

subject to:



*Pricing relationships for the OPF demand settings*

$$\lambda_P \frac{\partial P_s}{\partial P_i} + \lambda_Q \frac{\partial Q_s}{\partial P_i} + \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial P_i} + \sum_{k \in K} \eta_{Pk} \frac{\partial \bar{P}_k}{\partial P_i} + \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial P_i} + \beta_{Pi} = 0 \quad : P_{Di} \forall i \in PXS \quad (39)$$

$$\lambda_P \frac{\partial P_s}{\partial Q_i} + \lambda_Q \frac{\partial Q_s}{\partial Q_i} + \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial Q_i} + \sum_{k \in K} \eta_{Pk} \frac{\partial \bar{P}_k}{\partial Q_i} + \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial Q_i} + \beta_{Qi} = 0 \quad : Q_{Di} \forall i \in PXS \quad (40)$$

*Floor and ceiling constraints set by generator costs*

$$-\lambda_P \frac{\partial P_s}{\partial P_i} - \lambda_Q \frac{\partial Q_s}{\partial P_i} - \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial P_i} - \sum_{k \in K} \eta_{Pk} \frac{\partial \bar{P}_k}{\partial P_i} - \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial P_i} - v_{Pi}^+ \leq c_{Pi}^+ \quad : P_{Gi}^+ \forall i \in PXS \quad (41)$$

$$\lambda_P \frac{\partial P_s}{\partial P_i} + \lambda_Q \frac{\partial Q_s}{\partial P_i} + \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial P_i} + \sum_{k \in K} \eta_{Pk} \frac{\partial \bar{P}_k}{\partial P_i} + \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial P_i} - v_{Pi}^- \leq -c_{Pi}^+ \quad : P_{Gi}^- \forall i \in PXS \quad (42)$$

$$-\lambda_P \frac{\partial P_s}{\partial Q_i} - \lambda_Q \frac{\partial Q_s}{\partial Q_i} - \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial Q_i} - \sum_{k \in K} \eta_{Pk} \frac{\partial \bar{P}_k}{\partial Q_i} - \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial Q_i} - v_{Qi}^+ \leq c_{Qi}^+ \quad : Q_{Gi}^+ \forall i \in PXS \quad (43)$$

$$\lambda_P \frac{\partial P_s}{\partial Q_i} + \lambda_Q \frac{\partial Q_s}{\partial Q_i} + \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial Q_i} + \sum_{k \in K} \eta_{Pk} \frac{\partial \bar{P}_k}{\partial Q_i} + \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial Q_i} - v_{Qi}^- \leq -c_{Qi}^+ \quad : Q_{Gi}^- \forall i \in PXS \quad (44)$$

*Pricing relationships for the OPF transmission line constraints*

$$\eta_{Pk} - v_{Pk}^+ + v_{Pk}^- = 0 \quad : \bar{P}_k \forall k \in K \quad (45)$$

$$\eta_{Qk} - v_{Qk}^+ + v_{Qk}^- = 0 \quad : \bar{Q}_k \forall k \in K \quad (46)$$

*Pricing relationships for the OPF voltage constraints*

$$\mu_n - v_{Vn}^+ + v_{Vn}^- = 0 \quad : V_n \forall n \in PX \quad (47)$$

## SIMPLIFICATION OF THE DUAL

*The objective function*

For an optimal primal dispatch the primal linearised objective function value (20), will equal the optimal dual objective function value (38). This reflects the fact that the welfare of the system has been set by the primal dispatch solution, given the binding constraints, so any dual solution consistent with that dispatch must produce the same welfare. What may be desirable, though, is an objective function to *distribute* the wealth, in the event of degenerate dual solutions. Degeneracy can occur when the optimal solution lies on a corner of the generator cost curve, most probably because the load just happens to correspond to having the 'marginal' generator on full output. In that case any price between its marginal cost and that of the next most expensive generating unit may be compatible with the dispatch. It is debatable how often this will happen, and hence how significant these 'rent allocation' issues really are. However, the issue must be considered because there is wide commercial interest in the implication of the dual objective function in those instances where degeneracy does occur.

We observe that, since any feasible dual solution which obeys the complementary slackness conditions must be optimal, and since we can directly impose the complementary slackness conditions on the dual constraints simply by excluding all terms except those which correspond to binding primal constraints, we can make the natural dual objective function redundant in the sense that all remaining feasible solutions will be optimal with respect to that objective function. The choice of what objective function to use in the pricing model is then ultimately a policy issue.

Hogan (1991) observed that the dual objective function defines the rents, or profits, on all of the resources involved in the OPF problem. But, given our linearisation of the problem around the observed solution, this objective function will equal zero for an optimal dispatch, implying that the rent on all of the resources controlled by the grid company equals the negative of the rent on all the



resources owned by market players. This reasoning gives rise to Hogan's suggestion that by minimising the rent attributed to the network, we effectively maximise the rents on the resources owned by other parties. While this is likely to appeal to market participants, Read and Ring (1995) point out that maximising the rent to the network would actually provide a non-distortionary means of recovering some of the network fixed costs which will otherwise have to be recovered by other, more distortionary, means. In Hogan (1992) transmission congestion rents are redistributed to those who have paid for the network to provide a hedge against congestion costs in the form of transmission congestion contracts (San Diego, 1995). An alternative form of contract proposed by Read and Sell (1989) also returns loss rentals.

### The constraints

The constraints of the dual OPF can be simplified greatly. We first note that the swing bus power injections are dependent variables, so the derivatives involving these in (39) and (40) are all zero, with the exception of the derivatives of the injections themselves, which equal one. Hence, for the swing bus, these equations reduce to the form of (48) and (49).

$$\beta_{Ps} = \lambda_P \quad (48)$$

$$\beta_{Qs} = \lambda_Q \quad (49)$$

That is, the active and reactive power prices at the swing bus are set to the shadow prices of the energy conservation constraints, being the marginal values of active and reactive power supply there. These prices equal the price of producing power at the marginal bus(es) and delivering it to the swing bus. Commercially, it will probably be best to always have the system energy price defined at the same reference bus. Also, for reasons explained in Read and Ring (1995), it will be best, if using this  $PQ$  formulation, to always use the same swing bus, and to make this a generator bus. In practice then, it will often be advisable to make the swing bus the reference bus. Using (48) and (49) we can re-express (39) and (40) in the form of (51) and (52).

Substituting the active power prices for all the non-swing buses from (39) into (41) and (42), and the reactive power prices for all the same buses from (40) into (43) and (44), produces (53) and (54). We use ' $\langle z \rangle$ ' to denote that ' $z$ ' only appears in the dual formulation if the primal constraint to which it corresponds is binding. This same notation is used to translate (45)–(47) into (55)–(57). A rent minimising form of the dual objective, here assuming only active power is traded on a spot basis, is depicted in (50). If reactive power is also traded on a spot basis, then equivalent reactive power terms should be added as in Hogan (1991).

$$\begin{aligned} &\text{Maximise} \\ &\sum_{i \in PXS} \beta_{Pi} (P_{Gi}^* - P_{Di}^*) \\ &\text{subject to} \\ &\begin{aligned} &v_P^{+K}, v_P^{-K}, v_Q^{+K}, v_Q^{-K} \geq 0 \\ &v_P^{+PXS}, v_P^{-PXS}, v_Q^{+PXS}, v_Q^{-PXS} \geq 0 \\ &\beta_{Ps}, \beta_{Qs}, \eta_P^s, \eta_Q^s, \mu^{PX} \end{aligned} \end{aligned} \quad (50)$$

$$\beta_{Pi} = -\beta_{Ps} \frac{\partial P_s}{\partial P_i} - \beta_{Qs} \frac{\partial Q_s}{\partial P_i} - \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial P_i} - \sum_{k \in K} \eta_{pk} \frac{\partial \bar{P}_k}{\partial P_i} - \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial P_i} \quad \forall i \in PX \quad (51)$$

$$\beta_{Qi} = -\beta_{Ps} \frac{\partial P_s}{\partial Q_i} - \beta_{Qs} \frac{\partial Q_s}{\partial Q_i} - \sum_{n \in PX} \mu_n \frac{\partial V_n}{\partial Q_i} - \sum_{k \in K} \eta_{pk} \frac{\partial \bar{P}_k}{\partial Q_i} - \sum_{k \in K} \eta_{Qk} \frac{\partial \bar{Q}_k}{\partial Q_i} \quad \forall i \in PX \quad (52)$$

$$c_{Pi}^- - \langle v_{Pi}^- \rangle \leq \beta_{Pi} \leq c_{Pi}^+ + \langle v_{Pi}^+ \rangle \quad \forall i \in PXS \quad (53)$$

$$c_{Qi}^- - \langle v_{Qi}^- \rangle \leq \beta_{Qi} \leq c_{Qi}^+ + \langle v_{Qi}^+ \rangle \quad \forall i \in PXS \quad (54)$$

$$\mu_n = \langle v_{Vn}^+ \rangle - \langle v_{Vn}^- \rangle \quad \forall n \in PX \quad (55)$$

$$\eta_{pk} = \langle v_{Pk}^+ \rangle - \langle v_{Pk}^- \rangle \quad \forall k \in K \quad (56)$$

$$\eta_{Qk} = \langle v_{Qk}^+ \rangle - \langle v_{Qk}^- \rangle \quad \forall k \in K \quad (57)$$

$$\text{All } v \text{ terms non-negative.} \quad (58)$$

Note that the terms in (51) and (52) involve data readily available once the dispatch is known. Here



$\frac{\partial P_s}{\partial P_i}, \frac{\partial Q_s}{\partial P_i}, \frac{\partial P_s}{\partial Q_i}$ , and  $\frac{\partial Q_s}{\partial Q_i}$  describe the marginal generation and losses attributed to changes in the power flows. Similarly,  $\frac{\partial V_n}{\partial P_i}, \frac{\partial \bar{P}_k}{\partial P_i}, \frac{\partial \bar{Q}_k}{\partial P_i}, \frac{\partial V_n}{\partial Q_i}, \frac{\partial \bar{P}_k}{\partial Q_i}$ , and  $\frac{\partial \bar{Q}_k}{\partial Q_i}$  are the marginal impacts on the voltages and line flows, sometimes described by system operators as the dispatch 'shift factors'. To evaluate these derivatives we must apply a Jacobian coordinate transformation as the fundamental electrical equations only allow us to determine derivatives with respect to voltage magnitude and phase angle, as discussed by Hogan (1991) and Read and Ring (1995). In practice, this transformation amounts to solving a set of sparse linear equations, a problem which is straightforward to set up and solve.

The constraints of (51)–(58) have a nice mathematical interpretation. On the left of (51) and (52) we have all the prices for the independent dispatch variables, while on the right we have a linear combination of the prices corresponding to all the dependent dispatch variables. That is, the independent primal variables have dependent dual prices, and vice versa. While this result follows from standard duality theory, it has not widely been recognised in the spot pricing literature. Equations (53)–(57) give rise to another useful observation. If the primal quantity, say active power, is constrained then either one of the  $\langle v_{pi}^- \rangle$  or  $\langle v_{pi}^+ \rangle$  terms will be non-zero, allowing the price  $\beta_{pi}$  to lie outside of the range defined by the marginal costs  $c_{pi}^-$  and  $c_{pi}^+$ . Conversely, if active power is unconstrained then  $\langle v_{pi}^- \rangle = \langle v_{pi}^+ \rangle = 0$ , and the price must lie between  $c_{pi}^-$  and  $c_{pi}^+$ . This result holds for reactive power, voltage, and the transmission line flow variable as well, though in the latter two cases the marginal fuel costs are zero. Note that while transmission charges may vary over time, they only vary so as to reflect the real-time value of transmission to the system.

The sources of the marginal unit of active power, the marginal buses, will have  $\beta_{pi}$  strictly between  $c_{pi}^-$  and  $c_{pi}^+$  (which will generally be equal). If there are no constraints on the transmission of power there will be only one marginal bus, and the price at this bus will drive the price at all buses, including that at the swing bus. The effect of losses will determine the price at other points in the system relative to the swing bus. If a transmission constraint becomes binding then a lone marginal bus will not be able to supply power across that constraint, requiring the introduction of an additional marginal bus. The shadow price on the constrained line must take a value which explains the price difference between these two marginal buses.

In general, as extra constraints are introduced to the primal problem, extra price terms must be added to the dual if it is to be able to explain the effects of these constraints. These constraints might include contingency requirements, thermal power flow limits, or frequency control constraints. A major role of our model is to provide a mechanism by which dispatchers and policy makers can assess the often non-intuitive impact of such constraints.

On rare occasions, when the system is hard against its limits, the dual complementary slackness constraints may allow prices to be unbounded. In such cases, the choice of objective may make it possible, or impossible, to find a (finite) price solution. In practice, though, we can ensure that prices are always bounded above by the cost of non-supply, setting this as a price constraint on one or more buses. Prices are not likely to be unbounded in the other direction, although negative prices at some buses are possible when constraints occur in loops (Read and Ring, 1996).

In situations where the pricing problem is under-defined, which can occur if not all pricing constraints are modelled or if the marginal bus(es) have prices defined within a range of values, then the original dual objective function (38) cannot differentiate between the possible pricing solutions. This is where an arbitrary objective function, like that used in (50), has value. Alternatively, if the model is over-defined, due to shadow prices for actual primal constraints being excluded from the pricing model, a likely situation due to the complexity of power system operation, or due to primal sub-optimality, then there is no internally consistent set of prices compatible with the dispatch. In this situation the goal programming approach of Ring (1995) could be applied to the dual to determine the 'best compromise prices' which minimise the cost of deviations from consistent prices. This 'best compromise' pricing technique creates the potential to resolve pricing ambiguities caused by integer effects, and other modelling limitations of the approach.

With this model being focused on a snapshot view of the system, and given the unavoidable discrepancies between reality and the model, there is a need to clearly define the role and responsibility of the dispatcher. In particular, the ownership of the constrained resources, and the treatment of resources which are constrained over a short time horizon, but are unconstrained over a



longer time frame must be clarified. This task becomes more involved as market sophistication and requirements increase, and may place a limit on what can be implemented in practice.

## IMPLEMENTATION AND PERFORMANCE

### New Zealand

Trans Power New Zealand Limited operates a model based on the theory described above to determine half-hourly spot prices for each of the 600 buses of the New Zealand system. This model was prototyped using GAMS (Rosevear and Ring, 1992). By 1992 a model written in C had replaced GAMS. The C code calls CPLEX once to solve the set of linear equations which produce the required partial derivatives, treating the problem as a trivial linear program, and then again to solve the linear programming pricing problem. The resulting prices have been verified for unconstrained dispatches by comparing them with the change in system cost produced by individually perturbing the power injection at each bus using Power Technology, Inc.'s PSS/E power flow package.

Instead of the  $PQ$  representation described here, this model uses a more general  $PVQ$  representation (Read and Ring, 1995) which treats reactive power at some buses as dependent variables, making voltage magnitudes independent there, while the swing bus voltage is treated as an endogenous independent variable. So called ' $PI$ ' impedance models (Read et al., 1995), either individually or as sets, are used to represent 2 and 3 winding transformers, disconnected lines, condensers, capacitors and shunt devices. Terms can also be added to model additional constraints such as spinning reserve requirements (Ring et al., 1993).

The model, as currently implemented, assumes that there is no direct cost associated with reactive power, and can determine prices for active power, reactive power, voltage, spinning reserve target levels, and transmission line flow constraints. While it may be argued that, for an optimal dispatch, reactive power prices at most generator buses will be zero, this bound has not been enforced, so as to reduce the risk of infeasibility (reactive power prices are not currently charged to the market). A future version of the program will more strictly enforce these bounds.

The model produces prices, broken into component parts, corresponding to loss and constraint effects in approx. 90 s on a 486 PC. Including the set up time on a VAX mainframe and solution of the power flow the whole operation takes approx. 5 min.

At present, the commercial use of the model is limited to producing 'loss differentials', assuming an unconstrained network, for a representative sample of half hours in each year. These are then averaged, and applied to half-hourly energy prices to produce half-hourly spot prices for each bus in the system.

### United States

With the promise of simplified *ex post* pricing, major investigations have been underway in the United States to test the impacts of spot pricing and deal with a number of *ex post* pricing implementation issues. Here we summarise a few lessons in terms of the treatment of alternative constraint formulations, contingencies, aggregation, d.c.-load approximations and external transactions.

As in New Zealand, the  $PVQ$  formulation is the natural approach to use. Despite the common practice in OPF descriptions, however, the typical thermal limits on transmission lines do not take the form of (10) and (11). Rather, the thermal constraint is in terms of the MVA equivalent of current combining active and reactive power flows. Hence in a.c. applications the constraints in (10) and (11) are replaced by the single constraint  $\bar{P}_k^2 + \bar{Q}_k^2 \leq \text{MVA}^{\text{max}}$ . The corresponding changes apply to the resulting pricing problem. Again the necessary derivatives can be found from the solution of a sparse set of linear equations.

In applications involving contingency constraints, the corresponding elements for the derivative terms in (51) and (52) must be obtained for each binding contingency. Since this may involve a different configuration of the grid for each such contingency constraint, a different power flow and associated set of derivatives will be required. Again, calculation of these derivatives amounts to the



solution of a simple set of linear equations. Like the Trans Power model, a GAMS-OSL formulation solves the pricing problem in two stages. This model has been applied to obtain *ex post* prices on a network representation with 3768 buses. The present GAMS-OSL prototype implementation is slow, taking a few hours to complete a solution on a 486 PC. However, the great majority of the time is spent in the matrix generation phase, which is amenable to vast improvement with a specialised code as in the New Zealand experience.

The ability to solve the pricing problem, which is only a linear program, for a large network demonstrates the feasibility of *ex post* pricing. There is often an interest in avoiding the complexity of calculating prices for every location, preferring instead to aggregate to a relative small number of buses. The difficulty with such aggregation is that an exact aggregation of the underlying network is load and flow dependent. Hence, it is not possible to produce the parametric description of the network without knowing the solution. Given the computational feasibility of direct solution of the full network, however, it is actually easier to describe the network in terms of the actual detail, for which the network parameters are known *a priori* and then form the associated pricing problem. In practice, selecting representative buses is possible to reduce the information reported. The resulting working size of the pricing problem is, therefore, reduced to a number of rows approximately twice (for active and reactive terms) the number of reported buses plus the buses with price constraints, and the number of columns equal to the number of binding constraints in the dispatch. In even the 3768 bus case, therefore, the pricing problem might reduce (51) and (52) to a few hundred equations and a few columns for the binding flow or voltage constraints. The resulting pricing problem is trivial to solve, but invariably produces important information. For example, because of the interactions through the network, a single binding constraint is sufficient to produce different prices at every location.

There is a great interest in using d.c.-load approximations for the OPF problem. In effect, this takes the approximation in (20)–(37) and deletes all the reactive power and voltage terms or equations to obtain a reduced system in terms of active power only. The elimination of the reactive power components removes the most difficult non-linearity problems at the expense of foreclosing the ability to model voltage and reactive limitations. The benefit is the ability to solve very large primal problems. For example General Electric has a commercial code (MAPS-MWFLOW) that can solve a sophisticated inter-temporal, contingency constrained dispatch for a full 8760 hours of the year, running on advanced work stations in a few hours of elapsed time. As a proxy for voltage constraints, MAPS includes 'interface' limitations on the active power flows across certain transmission lines. In the usual way, this (large) linear program produces companion 'd.c.' prices that could be compared with the *ex post* a.c. prices. As discussed in Hogan (1993), these prices conform well in the presence of thermal constraints on lines. However, in the presence of voltage constraints at buses in the a.c. dispatch, the approximate interface constraints can lead to significant differences in the estimates of active power prices, especially for locations close to the bus with the voltage limitation. The problem is difficult to avoid for the simple reason that the true a.c. voltage constraint is highly non-linear and the d.c. interface approximation is set *a priori* as a simplified piecewise linear constraint. Without a good knowledge of the a.c. solution, it is difficult to define the appropriate linear approximation, even when the total active power flows across the interface might be approximately correct. It remains an open question as to which approximation would be acceptable for commercial purposes.

Finally, applications in the US confront the added problem of treating external transactions; power transactions that flow from outside the control of the system operator, and not subject to the same pricing regime. The system operator could model the full interconnected system (either exactly or approximately) or treat the net external injections as *PQ* buses disconnected from the rest of the external system. The pricing model can be adapted to either case. The prices calculated, however, will not in general be the same. In the case without any external network, under certain convexity assumptions we can show that the economic rents from congestion have a potentially useful revenue adequacy property that can always support congestion payments under a system of transmission congestion contracts (Hogan, 1992). This property does not hold in the case of external transactions if the external network is included in the model in calculating *ex post* prices that then apply only to transactions under the domain of the system operator. However, treating the points of connection as fixed loads, and deleting the external links, returns the pricing model to the canonical form and does produce a set of prices that are revenue adequate in this sense.



## CONCLUSIONS

We have presented a methodology for determining spatially varying spot prices given an observed power system dispatch. This approach extends previous work by allowing a wide range of complex a.c. power system phenomena to be accounted for. While the resulting pricing equations appear intricate, we have shown that they have a simple well defined underlying form.

This work highlights the contribution that duality theory can make to policy research. We have shown that simply forming the dual of a standard engineering OPF formulation, and applying a number of simplifications arising from duality theory, we can deduce an economically consistent and commercially useful tool for valuing, coordinating and studying transactions in an electricity market. This type of approach is one which we believe should be practised more in the Operations Research community.

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## APPENDIX

The constant terms in the linearised primal are defined as follows:

$$A_1^* = P_s^* - \sum_{i \in PX} \frac{\partial P_s}{\partial P_i} P_{Di}^* - \sum_{i \in PX} \frac{\partial P_s}{\partial Q_i} Q_{Di}^*$$

$$A_2^* = Q_s^* - \sum_{i \in PX} \frac{\partial Q_s}{\partial P_i} P_{Di}^* - \sum_{i \in PX} \frac{\partial Q_s}{\partial Q_i} Q_{Di}^*$$

$$A_{3n}^* = V_n^* - \sum_{i \in PX} \frac{\partial V_n}{\partial P_i} P_{Di}^* - \sum_{i \in PX} \frac{\partial V_n}{\partial Q_i} Q_{Di}^* \quad \forall n \in PX$$

$$A_4^* = \bar{P}_k^* - \sum_{i \in PX} \frac{\partial \bar{P}_k}{\partial P_i} P_{Di}^* - \sum_{i \in PX} \frac{\partial \bar{P}_k}{\partial Q_i} Q_{Di}^* \quad \forall k \in K$$

$$A_{5k}^* = \bar{Q}_k^* - \sum_{i \in PX} \frac{\partial \bar{Q}_k}{\partial P_i} P_{Di}^* - \sum_{i \in PX} \frac{\partial \bar{Q}_k}{\partial Q_i} Q_{Di}^* \quad \forall k \in K$$



# APPENDIX

The constant terms in the linearized program are defined as follows:

$$A1 = P1 - \sum_{i=1}^n \frac{P1_i}{Q1_i} P1_i - \sum_{i=1}^n \frac{P1_i}{Q1_i} Q1_i$$

$$A2 = Q1 - \sum_{i=1}^n \frac{Q1_i}{P1_i} P1_i - \sum_{i=1}^n \frac{Q1_i}{P1_i} Q1_i$$

$$A3 = P1 - \sum_{i=1}^n \frac{P1_i}{Q1_i} P1_i - \sum_{i=1}^n \frac{P1_i}{Q1_i} Q1_i$$

$$A4 = P1 - \sum_{i=1}^n \frac{P1_i}{Q1_i} P1_i - \sum_{i=1}^n \frac{P1_i}{Q1_i} Q1_i$$

$$A5 = Q1 - \sum_{i=1}^n \frac{Q1_i}{P1_i} P1_i - \sum_{i=1}^n \frac{Q1_i}{P1_i} Q1_i$$